ACE
Engineering College

DEPARTMENT OF MECHANICAL ENGINEERING

STRENGTH OF MATERIALS

LECTURE 1

INTRODUCTION AND REVIEW

Engineering science is usually subdivided into number of topics such as

1. Solid Mechanics

2. Fluid Mechanics

3. Heat Transfer

4. Properties of materials and soon Although there are close links between them in terms of the physical principles involved and methods of analysis employed.

The solid mechanics as a subject may be defined as a branch of applied mechanics that deals with behaviours of solid bodies subjected to various types of loadings. This is usually subdivided into further two streams i.e Mechanics of rigid bodies or simply Mechanics and Mechanics of deformable solids.

The mechanics of deformable solids which is branch of applied mechanics is known by several names i.e. strength of materials, mechanics of materials etc.

Mechanics of rigid bodies:

The mechanics of rigid bodies is primarily concerned with the static and dynamic behaviour under external forces of engineering components and systems which are treated as infinitely strong and undeformable Primarily we deal here with the forces and motions associated with particles and rigid bodies.

Mechanics of deformable solids :

Mechanics of solids:

The mechanics of deformable solids is more concerned with the internal forces and associated changes in the geometry of the components involved. Of particular importance are the properties of the materials used, the strength of which will determine whether the components fail by breaking in service, and the stiffness of which will determine whether the amount of deformation they suffer is acceptable. Therefore, the subject of mechanics of materials or strength of materials is central to the whole activity of engineering design. Usually the objectives in analysis here will be the determination of the stresses, strains, and deflections produced by loads. Theoretical analyses and experimental results have an equal roles in this field.

Analysis of stress and strain :

Concept of stress: Let us introduce the concept of stress as we know that the main problem of engineering mechanics of material is the investigation of the internal resistance of the body, i.e. the nature of forces set up within a body to balance the effect of the externally applied forces.

The externally applied forces are termed as loads. These externally applied forces may be due to any one of the reason.

(i) due to service conditions

(ii) due to environment in which the component works

(iii) through contact with other members

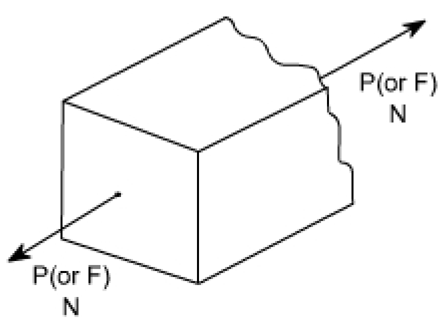
(iv) due to fluid pressures

(v) due to gravity or inertia forces.

As we know that in mechanics of deformable solids, externally applied forces acts on a body and body suffers a deformation. From equilibrium point of view, this action should be opposed or reacted by internal forces which are set up within the particles of material due to cohesion.

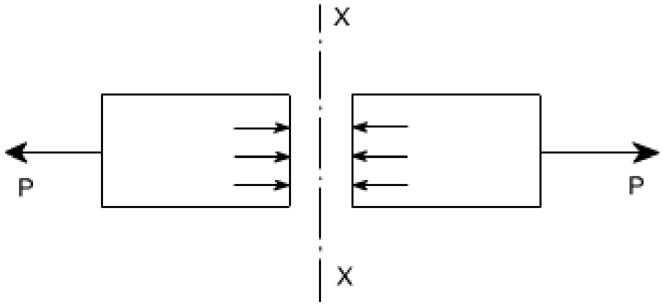
These internal forces give rise to a concept of stress. Therefore, let us define a stress Therefore, let us define a term stress

Stress:



Let us consider a rectangular bar of some cross – sectional area and subjected to some load or force (in Newtons)

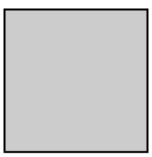
Let us imagine that the same rectangular bar is assumed to be cut into two halves at section XX. The each portion of this rectangular bar is in equilibrium under the action of load P and the internal forces acting at the section XX has been shown



Now stress is defined as the force intensity or force per unit area. Here we use a symbol s to represent the stress.

$$\sigma = \frac{P}{A}$$

Where A is the area of the X – section



Here we are using an assumption that the total force or total load carried by the rectangular bar is uniformly distributed over its cross – section.

But the stress distributions may be for from uniform, with local regions of high stress known as stress concentrations.

If the force carried by a component is not uniformly distributed over its cross – sectional area, A, we must consider a small area, ‘dA' which carries a small load dP, of the total force ‘P', Then definition of stress is

$$\boxed{\sigma = \frac{\delta F}{\delta A}}$$

As a particular stress generally holds true only at a point, therefore it is defined mathematically as

$$\boxed{\sigma = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}}$$

Units :

The basic units of stress in S.I units i.e. (International system) are N / m^2 (or Pa)

MPa = 10^6 Pa

GPa = 10^9 Pa

KPa = 10^3 Pa

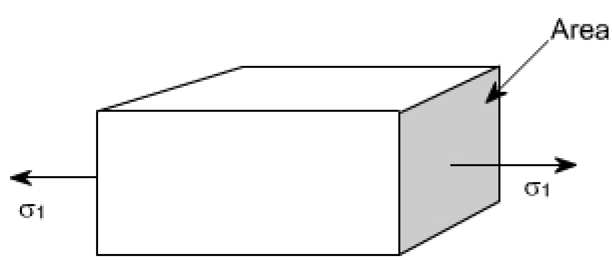
Some times N / mm^2 units are also used, because this is an equivalent to MPa. While US customary unit is pound per square inch psi.

TYPES OF STRESSES :

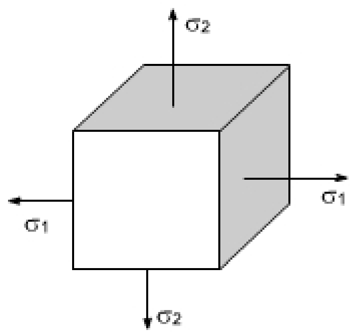
only two basic stresses exists : (1) normal stress and (2) shear shear stress. Other stresses either are similar to these basic stresses or are a combination of these e.g. bending stress is a combination tensile, compressive and shear stresses. Torsional stress, as encountered in twisting of a shaft is a shearing stress.

Let us define the normal stresses and shear stresses in the following sections.

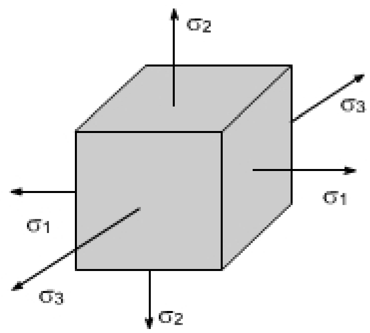
Normal stresses : We have defined stress as force per unit area. If the stresses are normal to the areas concerned, then these are termed as normal stresses. The normal stresses are generally denoted by a Greek letter (s)



This is also known as uniaxial state of stress, because the stresses acts only in one direction however, such a state rarely exists, therefore we have biaxial and triaxial state of stresses where either the two mutually perpendicular normal stresses acts or three mutually perpendicular normal stresses acts as shown in the figures below :



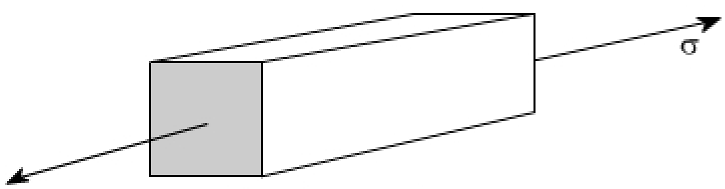
(Biaxial state of stress)



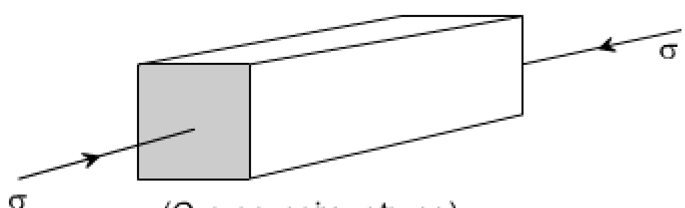
(Triaxial state of stress)

Tensile or compressive stresses :

The normal stresses can be either tensile or compressive whether the stresses acts out of the area or into the area

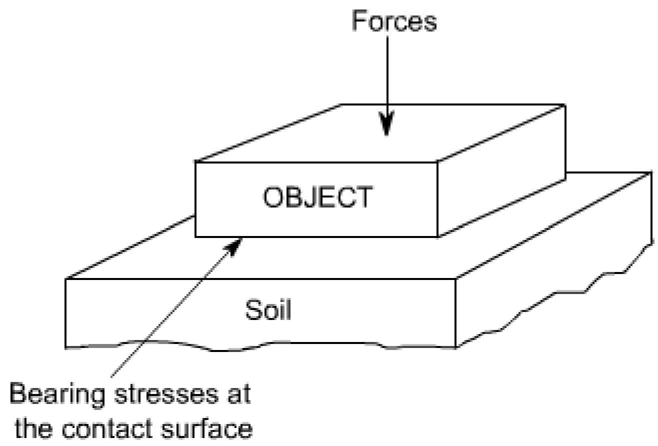


(Tensile stress)



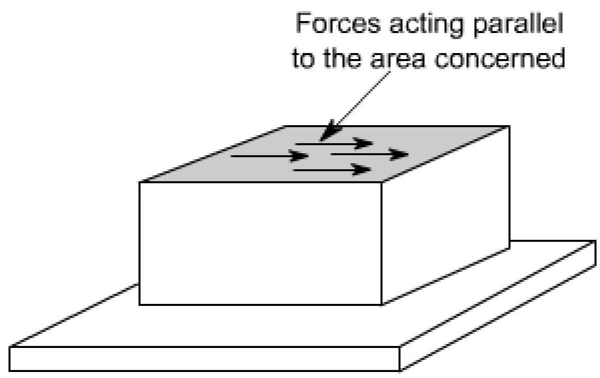
(Compressive stress)

Bearing Stress : When one object presses against another, it is referred to a bearing stress (They are in fact the compressive stresses).



Shear stresses :

Let us consider now the situation, where the cross – sectional area of a block of material is subject to a distribution of forces which are parallel, rather than normal, to the area concerned. Such forces are associated with a shearing of the material, and are referred to as shear forces. The resulting force interistes are known as shear stresses.



The resulting force intensities are known as shear stresses, the mean shear stress being equal to

$$\tau = \frac{P}{A}$$

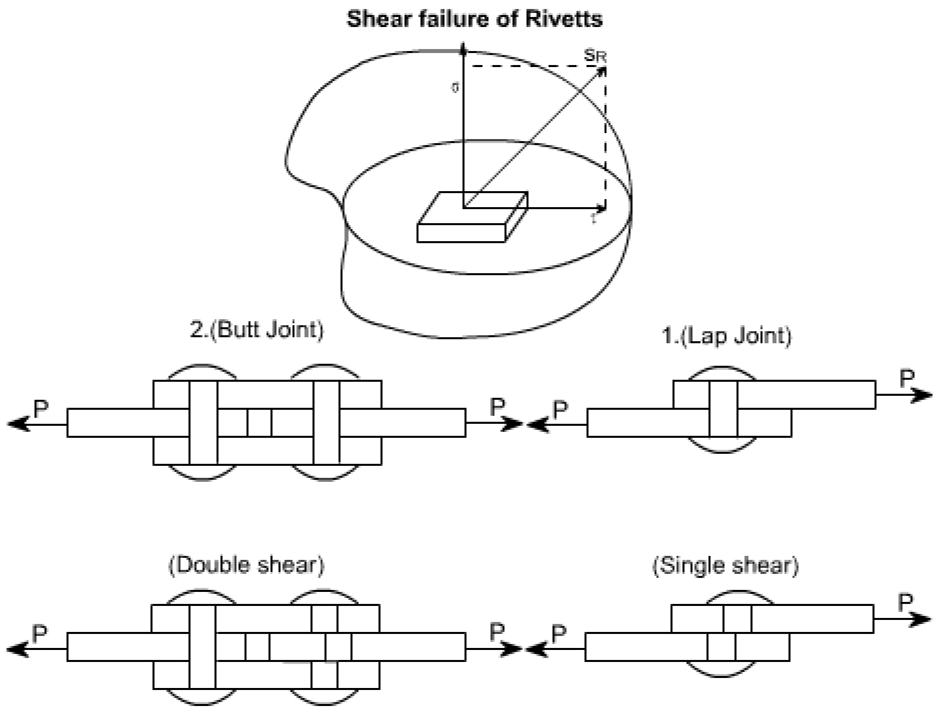
Where P is the total force and A the area over which it acts.

As we know that the particular stress generally holds good only at a point therefore we can define shear stress at a point as

$$\tau = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}$$

The greek symbol t (tau) (suggesting tangential) is used to denote shear stress.

However, it must be borne in mind that the stress (resultant stress) at any point in a body is basically resolved into two components s and t one acts perpendicular and other parallel to the area concerned, as it is clearly defined in the following figure.



The single shear takes place on the single plane and the shear area is the cross - sectional of the rivett, whereas the double shear takes place in the case of Butt joints of rivetts and the shear area is the twice of the X - sectional area of the rivett.

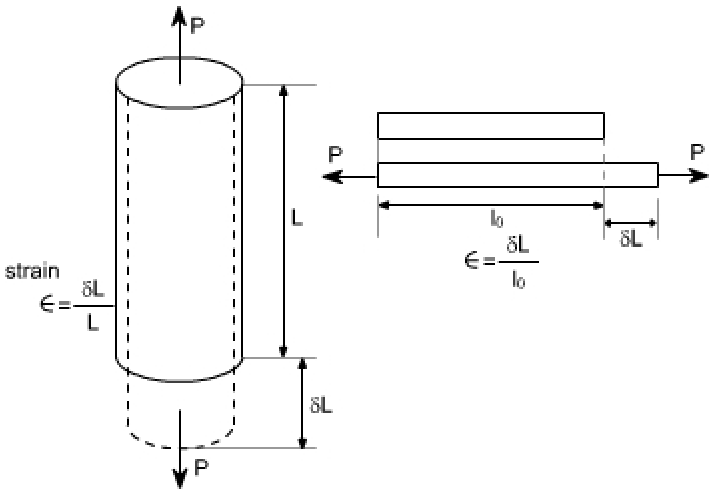
ANALYSIS OF STRAINS

CONCEPT OF STRAIN

Concept of strain: if a bar is subjected to a direct load, and hence a stress the bar will change in length. If the bar has an original length L and changes by an amount dL, the strain produce is defined as follows:

$$\text{strain}(\varepsilon) = \frac{\text{change in length}}{\text{orginal length}} = \frac{\delta L}{L}$$

Strain is thus, a measure of the deformation of the material and is a no dimensional Quantity i.e. it has no units. It is simply a ratio of two quantities with the same unit.



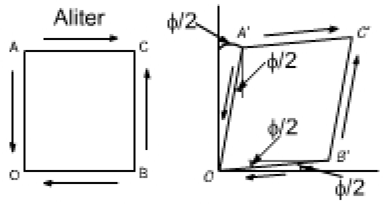
Since in practice, the extensions of materials under load are very very small, it is often convenient to measure the strain in the form of strain x 10^{-6} i.e. micro strain, when the symbol used becomes m Î.

Sign convention for strain:

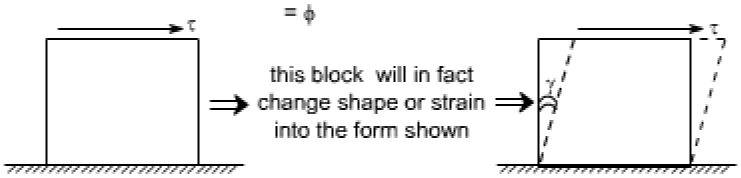
Tensile strains are positive whereas compressive strains are negative. The strain defined earlier was known as linear strain or normal strain or the longitudinal strain now let us defines the shear strain.

Definition: An element which is subjected to a shear stress experiences a deformation as shown in the figure below. The tangent of the angle through which two adjacent sides rotate relative to their initial position is termed shear strain. In many cases the angle is very small and the angle itself is used, (in radians), instead of tangent, so that g = Ð AOB - Ð A'OB' = f

Shear strain: As we know that the shear stresses acts along the surface. The action of the stresses is to produce or being about the deformation in the body consider the distortion produced b shear sheer stress on an element or rectangular block



$$\gamma = \angle AOB - \angle A'OB' = \phi$$



This shear strain or slide is f and can be defined as the change in right angle. Or the angle of deformation g is then termed as the shear strain. Shear strain is measured in radians & hence is non – dimensional i.e. it has no unit.So we have two types of strain i.e. normal stress & shear stresses.

Hook's Law:

A material is said to be elastic if it returns to its original, unloaded dimensions when load is removed.

Hook's law therefore states that

Stress (s) a strain (Î)

$$\boxed{\frac{\text{stress}}{\text{strain}} = \text{constant}}$$

Modulus of elasticity: Within the elastic limits of materials i.e. within the limits in which Hook's law applies, it has been shown that

Stress / strain = constant

This constant is given by the symbol E and is termed as the modulus of elasticity or Young's modulus of elasticity

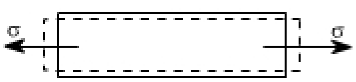
$$E = \frac{\text{strain}}{\text{stress}} = \frac{\sigma}{\epsilon}$$

$$= P/A \Big/ \delta L/L$$

Thus $\boxed{E = \frac{PL}{A\delta L}}$

The value of Young's modulus E is generally assumed to be the same in tension or compression and for most engineering material has high, numerical value of the order of 200 GPa

Poisson's ratio: If a bar is subjected to a longitudinal stress there will be a strain in this direction equal to s / E . There will also be a strain in all directions at right angles to s . The final shape being shown by the dotted lines.

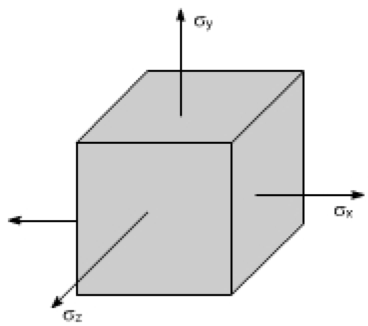


It has been observed that for an elastic material, the lateral strain is proportional to the longitudinal strain. The ratio of the lateral strain to longitudinal strain is known as the poison's ratio.

Poison's ratio (m) = - lateral strain / longitudinal strain

For most engineering materials the value of m his between 0.25 and 0.33.

Three – dimensional state of strain: Consider an element subjected to three mutually perpendicular tensile stresses s_x, $s_y$and s_z as shown in the figure below.



If s_y and s_z were not present the strain in the x direction from the basic definition of Young's modulus of Elasticity E would be equal to

Î$_x$= s_x/ E

The effects of s_y and s_z in x direction are given by the definition of Poisson's ratio ‘ m ' to be equal as -m s_y/ E and -m s_z/ E

The negative sign indicating that if $s_y$and s_z are positive i.e. tensile, these they tend to reduce the strain in x direction thus the total linear strain is x direction is given by

$$\epsilon_x = \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_z}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E} - \mu\frac{\sigma_z}{E}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E}$$

Principal strains in terms of stress:

In the absence of shear stresses on the faces of the elements let us say that s_x , s_y , s_z are in fact the principal stress. The resulting strain in the three directions would be the principal strains.

i.e. We will have the following relation.

$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu\sigma_2 - \mu\sigma_3\right]$$

$$\epsilon_2 = \frac{1}{E}\left[\sigma_2 - \mu\sigma_1 - \mu\sigma_3\right]$$

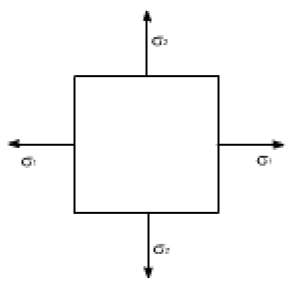
$$\epsilon_3 = \frac{1}{E}\left[\sigma_3 - \mu\sigma_1 - \mu\sigma_2\right]$$

For Two dimensional strain: system, the stress in the third direction becomes zero i.e $s_z = 0$ or $s_3 = 0$

Although we will have a strain in this direction owing to stresses s_1& s_2 .

Hence the set of equation as described earlier reduces to

$$\epsilon_1 = \frac{1}{E}[\sigma_1 - \mu\sigma_2]$$
$$\epsilon_2 = \frac{1}{E}[\sigma_2 - \mu\sigma_1]$$
$$\epsilon_3 = \frac{1}{E}[-\mu\sigma_1 - \mu\sigma_2]$$



Hence a strain can exist without a stress in that direction

i.e if $\sigma_3 = 0;\ \epsilon_3 = \frac{1}{E}[-\mu\sigma_1 - \mu\sigma_2]$

Also

$\epsilon_1 .E = \sigma_1 - \mu\sigma_2$

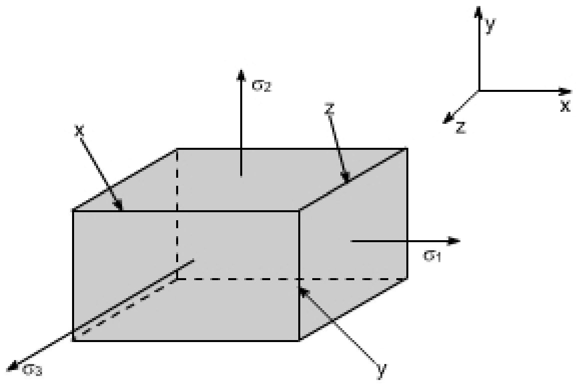
$\epsilon_2 .E = \sigma_2 - \mu\sigma_1$

so the solution of above two equations yields

$$\sigma_1 = \frac{E}{(1-\mu^2)}[\epsilon_1 + \mu\epsilon_2]$$
$$\sigma_2 = \frac{E}{(1-\mu^2)}[\epsilon_2 + \mu\epsilon_1]$$

Hydrostatic stress : The term Hydrostatic stress is used to describe a state of tensile or compressive stress equal in all directions within or external to a body. Hydrostatic stress causes a change in volume of a material, which if expressed per unit of original volume gives a volumetric strain denoted by $Î_v$. So let us determine the expression for the volumetric strain.

Volumetric Strain:



Consider a rectangle solid of sides x, y and z under the action of principal stresses s_1 , s_2 , s_3 respectively.

Then Î$_1$, Î$_2$, and Î$_3$ are the corresponding linear strains, than the dimensions of the rectangle becomes

(x + Î$_1$. x); (y + Î$_2$. y); (z + Î$_3$. z)

hence

$$\text{Volumetric strain} = \frac{\text{Increase in volume}}{\text{Original volume}}$$

$$= \frac{x(1+\epsilon_1)y(1+\epsilon_2)(1+\epsilon_3)z - xyz}{xyz}$$

the

$$= (1+\epsilon_1)y(1+\epsilon_2)(1+\epsilon_3) - 1 \cong \epsilon_1 + \epsilon_2 + \epsilon_3 \left[\text{Neglecting the products of } \epsilon\text{'s}\right]$$

ALITER : Let a cuboid of material having initial sides of Length x, y and z. If under some load system, the sides changes in length by dx, dy, and dz then the new volume (x + dx) (y + dy) (z +dz)

New volume = xyz + yzdx + xzdy + xydz

Original volume = xyz

Change in volume = yzdx +xzdy + xydz

Volumetric strain = (yzdx +xzdy + xydz) / xyz = Î$_x$+ Î$_y$+ Î$_z$

Neglecting the products of epsilon's since the strains are sufficiently small.

Volumetric strains in terms of principal stresses:

As we know that

$$\epsilon_1 = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E}$$

$$\epsilon_2 = \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_3}{E}$$

$$\epsilon_3 = \frac{\sigma_3}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$$

Futher Volumetric strain $= \epsilon_1 + \epsilon_2 + \epsilon_3$

$$= \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{E} - \frac{2\mu(\sigma_1 + \sigma_2 + \sigma_3)}{E}$$

$$= \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\mu)}{E}$$

hence the

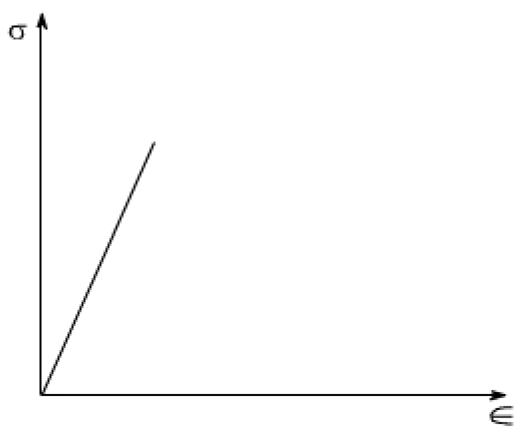
$$\boxed{\text{Volumetric strain} = \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\mu)}{E}}$$

STRESS - STRAIN RELATIONS

Stress – Strain Relations: The Hook's law, states that within the elastic limits the stress is proportional to the strain since for most materials it is impossible to describe the entire stress – strain curve with simple mathematical expression, in any given problem the behavior of the materials is represented by an idealized stress – strain curve, which emphasizes those aspects of the behaviors which are most important is that particular problem.

(i) Linear elastic material:

A linear elastic material is one in which the strain is proportional to stress as shown below:

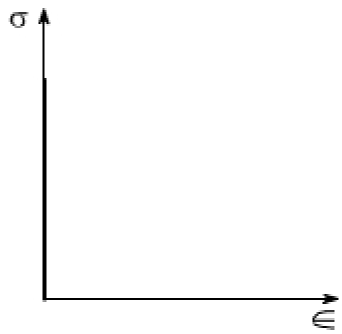


Linearly elastic material

There are also other types of idealized models of material behavior.

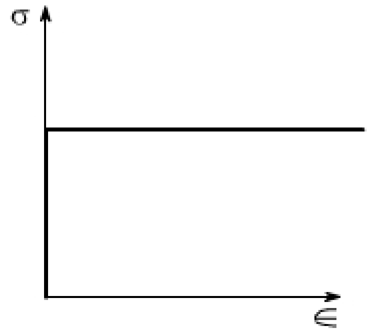
(ii) Rigid Materials:

It is the one which donot experience any strain regardless of the applied stress.



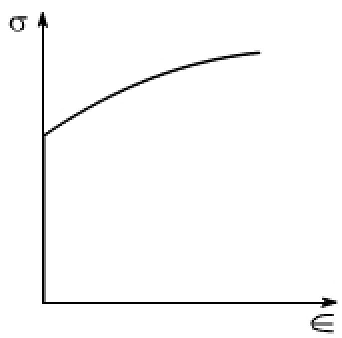
(iii) Perfectly plastic(non-strain hardening):

A perfectly plastic i.e non-strain hardening material is shown below:



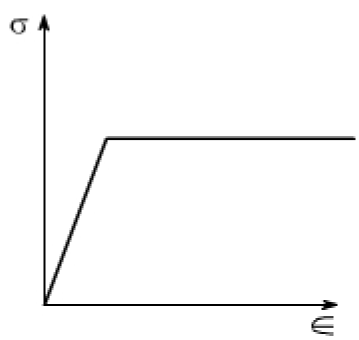
(iv) Rigid Plastic material(strain hardening):

A rigid plastic material i.e strain hardening is depicted in the figure below:



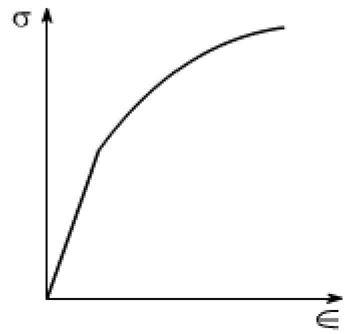
(v) Elastic Perfectly Plastic material:

The elastic perfectly plastic material is having the characteristics as shown below:



(vi) Elastic – Plastic material:

The elastic plastic material exhibits a stress Vs strain diagram as depicted in the figure below:



Elastic Stress – strain Relations :

Previously stress – strain relations were considered for the special case of a uniaxial loading i.e. only one component of stress i.e. the axial or normal component of stress was coming into picture. In this section we shall generalize the elastic behavior, so as to arrive at the relations which connect all the six components of stress with the six components of elastic stress. Futher, we would restrict overselves to linearly elastic material.

Before writing down the relations let us introduce a term ISOTROPY

ISOTROPIC: If the response of the material is independent of the orientation of the load axis of the sample, then we say that the material is isotropic or in other words we can say that isotropy of a material in a characteristics, which gives us the information that the properties are the same in the three orthogonal directions x y z, on the other hand if the response is dependent on orientation it is known as anisotropic.

Examples of anisotropic materials, whose properties are different in different directions are

(i) Wood

(ii) Fibre reinforced plastic

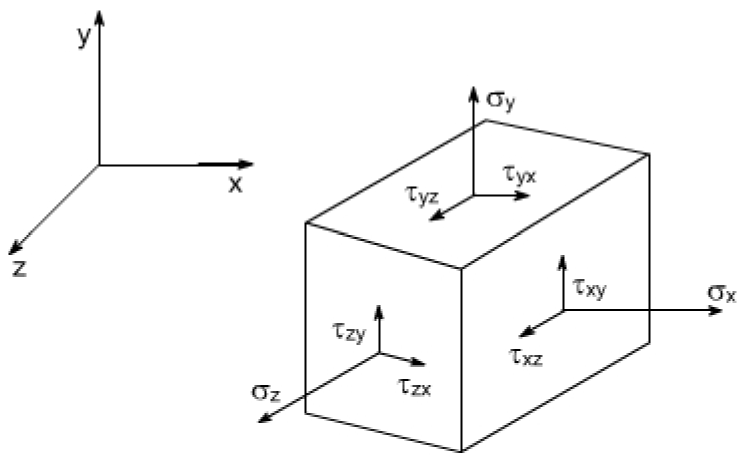
(iii) Reinforced concrete

HOMOGENIUS: A material is homogenous if it has the same composition through our body. Hence the elastic properties are the same at every point in the body. However, the properties need not to be the same in all the direction for the material to be homogenous. Isotropic materials have the same elastic properties in all the directions. Therefore, the material must be both homogenous and isotropic in order to have the lateral strains to be same at every point in a particular component.

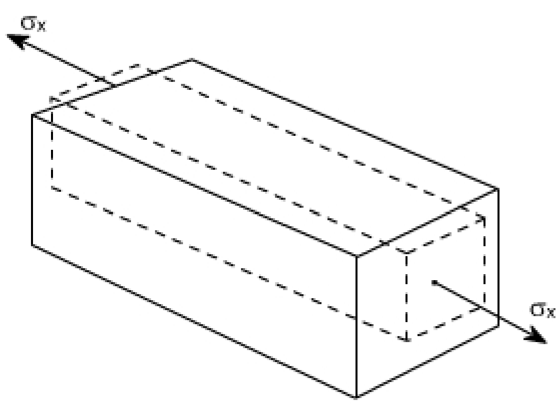
Generalized Hook's Law: We know that for stresses not greater than the proportional limit.

$$\epsilon = \frac{\sigma}{E} \text{ or } \mu = -\frac{|\epsilon_{lateral}|}{|\epsilon_{axial}|}$$

These equation expresses the relationship between stress and strain (Hook's law) for uniaxial state of stress only when the stress is not greater than the proportional limit. In order to analyze the deformational effects produced by all the stresses, we shall consider the effects of one axial stress at a time. Since we presumably are dealing with strains of the order of one percent or less. These effects can be superimposed arbitrarily. The figure below shows the general triaxial state of stress.



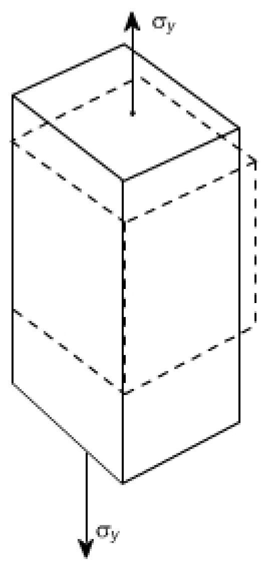
Let us consider a case when s_x alone is acting. It will cause an increase in dimension in X-direction whereas the dimensions in y and z direction will be decreased.



$$\epsilon_x = \frac{\sigma_x}{E}, \epsilon_y = -\mu\epsilon_x ; \epsilon_z = -\mu\epsilon_x$$

Therefore the resulting strains in three directions are $\epsilon_x = \frac{\sigma_x}{E} ; \epsilon_y = -\mu\frac{\sigma_x}{E} ; \epsilon_z = -\mu\frac{\sigma_x}{E}$

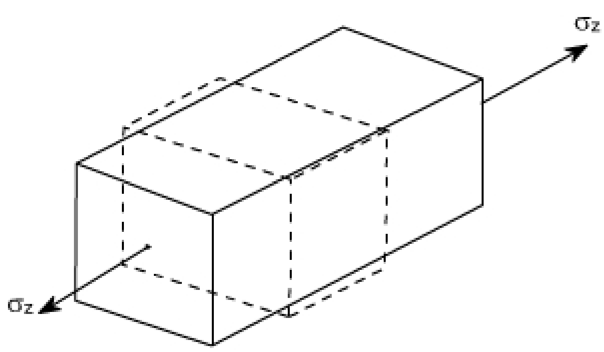
Similarly let us consider that normal stress s_y alone is acting and the resulting strains are



$$\epsilon_y = \frac{\sigma_y}{E}\ ,\ \epsilon_x = -\mu\,\epsilon_y\ ;\ \epsilon_z = -\mu\,\epsilon_y$$

$$\epsilon_y = \frac{\sigma_y}{E}\ ;\ \epsilon_x = -\mu\frac{\sigma_y}{E}\ ;\ \epsilon_z = -\mu\frac{\sigma_y}{E}$$

Now let us consider the stress s_z acting alone, thus the strains produced are



$$\epsilon_z = \frac{\sigma_z}{E}\ ,\ \epsilon_y = -\mu\,\epsilon_z\ ;\ \epsilon_x = -\mu\,\epsilon_z$$

$$\epsilon_z = \frac{\sigma_z}{E}\ ;\ \epsilon_y = -\mu\frac{\sigma_z}{E}\ ;\ \epsilon_x = -\mu\frac{\sigma_z}{E}$$

Thus the total strain in any one direction is

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\mu}{E}(\sigma_y + \sigma_z) \qquad (1)$$

In a similar manner, the total strain in the y and z directions become

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\mu}{E}(\sigma_x + \sigma_z) \qquad (2)$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\mu}{E}(\sigma_x + \sigma_y) \qquad (3)$$

In the following analysis shear stresses were not considered. It can be shown that for an isotropic material's a shear stress will produce only its corresponding shear strain and will not influence the axial strain. Thus, we can write Hook's law for the individual shear

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad (4)$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} \qquad (5)$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} \qquad (6)$$

Strains and shear stresses in the following manner.

The Equations (1) through (6) are known as Generalized Hook's law and are the constitutive equations for the linear elastic isotropic materials. When these equations isotropic materials. When these equations are used as written, the strains can be completely determined from known values of the stresses. To engineers the plane stress situation is of much relevance (i.e. $s_z = t_{xz} = t_{yz} = 0$), Thus then the above set of equations reduces to

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\mu\sigma_x}{E}$$

$$\epsilon_z = -\mu\frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} \text{ and } \tau_{xy} = \frac{\gamma_{xy}}{G}$$

Their inverse relations can be also determined and are given as

$$\sigma_x = \frac{E}{(1-\mu^2)}(\epsilon_x + \mu\,\epsilon_y)$$

$$\sigma_y = \frac{E}{(1-\mu^2)}(\epsilon_y + \mu\,\epsilon_x)$$

$$\tau_{xy} = G.\gamma_{xy}$$

Hook's law is probably the most well known and widely used constitutive equations for an engineering material." However, we cannot say that all the engineering materials are linear elastic isotropic ones. Because now in the present times, the new materials are being developed every day. Many useful materials exhibit nonlinear response and are not elastic too.

ELASTIC CONSTANTS

In considering the elastic behavior of an isotropic materials under, normal, shear and hydrostatic loading, we introduce a total of four elastic constants namely E, G, K, and g .

It turns out that not all of these are independent to the others. In fact, given any two of them, the other two can be foundout . Let us define these elastic constants

(i) E = Young's Modulus of Rigidity
= Stress / strain

(ii) G = Shear Modulus or Modulus of rigidity

= Shear stress / Shear strain

(iii) g = Poisson's ratio

= - lateral strain / longitudinal strain

(iv) K = Bulk Modulus of elasticity

= Volumetric stress / Volumetric strain

Where

Volumetric strain = sum of linear stress in x, y and z direction.

Volumetric stress = stress which cause the change in volume.

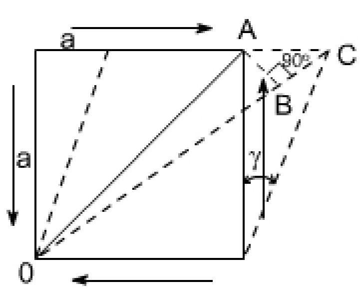
Let us find the relations between them

RELATION AMONG ELASTIC CONSTANTS

Relation between E, G and u :

Let us establish a relation among the elastic constants E,G and u. Consider a cube of material of side 'a' subjected to the action of the shear and complementary shear stresses as shown in the figure and producing the strained shape as shown in the figure below.

Assuming that the strains are small and the angle A C B may be taken as 45^0.



Therefore strain on the diagonal OA

= Change in length / original length

Since angle between OA and OB is very small hence OA @ OB therefore BC, is the change in the length of the diagonal OA

Thus, strain on diagonal OA = $\dfrac{BC}{OA}$

$$= \frac{AC\cos 45^0}{OA}$$

$$OA = \frac{a}{\sin 45^0} = a.\sqrt{2}$$

hence

$$\text{strain} = \frac{AC}{a\sqrt{2}}.\frac{1}{\sqrt{2}}$$

$$= \frac{AC}{2a}$$

but AC = $a\gamma$

where γ = shear strain

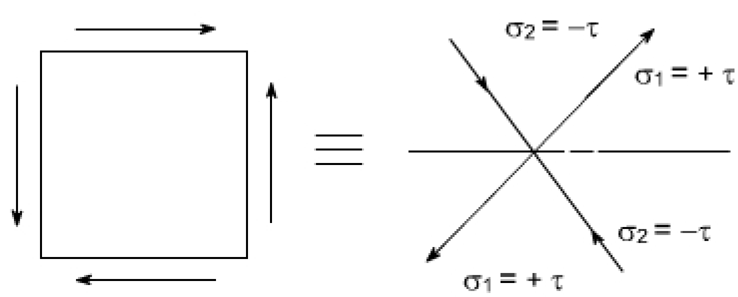
Thus, the strain on diagonal = $\dfrac{a\gamma}{2a} = \dfrac{\gamma}{2}$

From the definition

$$G = \frac{\tau}{\gamma} \text{ or } \gamma = \frac{\tau}{G}$$

thus, the strain on diagonal = $\dfrac{\gamma}{2} = \dfrac{\tau}{2G}$

Now this shear stress system is equivalent or can be replaced by a system of direct stresses at 45^0 as shown below. One set will be compressive, the other tensile, and both will be equal in value to the applied shear strain.



Thus, for the direct state of stress system which applies along the diagonals:

$$\text{strain on diagonal} = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$$

$$= \frac{\tau}{E} - \mu\frac{(-\tau)}{E}$$

$$= \frac{\tau}{E}(1 + \mu)$$

equating the two strains one may get

$$\frac{\tau}{2G} = \frac{\tau}{E}(1 + \mu)$$

or

$$\boxed{E = 2G(1 + \mu)}$$

We have introduced a total of four elastic constants, i.e E, G, K and g. It turns out that not all of these are independent of the others. In fact given any two of them, the other two can be found.

Again $E = 3K(1 - 2\gamma)$

$$\Rightarrow \frac{E}{3(1-2\gamma)} = K$$

if $\gamma = 0.5$ $K = \infty$

$$\epsilon_v = \frac{(1-2\gamma)}{E}(\epsilon_x + \epsilon_y + \epsilon_z) = 3\frac{\sigma}{E}(1-2\gamma)$$

(for $\epsilon_x = \epsilon_y = \epsilon_z$ hydrostatic state of stress)

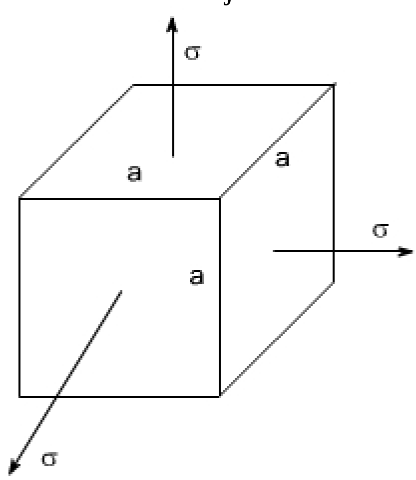
$\epsilon_v = 0$ if $\gamma = 0.5$

Irrespective of the stresses i.e., the material is incompressible.

When g = 0.5 Value of k is infinite, rather than a zero value of E and volumetric strain is zero, or in other words, the material is incompressible.

Relation between E, K and u :

Consider a cube subjected to three equal stresses s as shown in the figure below



The total strain in one direction or along one edge due to the application of hydrostatic stress or volumetric stress s is given as

$$= \frac{\sigma}{E} - \gamma\frac{\sigma}{E} - \gamma\frac{\sigma}{E}$$

$$= \frac{\sigma}{E}(1 - 2\gamma)$$

volumetre strain = 3.linear strain

volumetre strain = $\epsilon_x + \epsilon_y + \epsilon_z$

or thus, $\epsilon_x = \epsilon_y = \epsilon_z$

$$\text{volumetric strain} = 3\frac{\sigma}{E}(1 - 2\gamma)$$

By definition

$$\text{Bulk Modulus of Elasticity (K)} = \frac{\text{Volumetric stress}(\sigma)}{\text{Volumetric strain}}$$

or

$$\text{Volumetric strain} = \frac{\sigma}{k}$$

Equating the two strains we get

$$\frac{\sigma}{k} = 3.\frac{\sigma}{E}(1 - 2\gamma)$$

$$\boxed{E = 3K(1 - 2\gamma)}$$

Relation between E, G and K :

The relationship between E, G and K can be easily determined by eliminating u from the already derived relations

E = 2 G (1 + u) and E = 3 K (1 - u)

Thus, the following relationship may be obtained

$$\boxed{E = \frac{9\,GK}{(3K + G)}}$$

Relation between E, K and g :

From the already derived relations, E can be eliminated

$$E = 2G(1+\gamma)$$

$$E = 3K(1-2\gamma)$$

Thus,we get

$$3k(1-2\gamma) = 2G(1+\gamma)$$

therefore

$$\gamma = \frac{(3K-2G)}{2(G+3K)}$$

or

$$\boxed{\gamma = 0.5(3K-2G)(G+3K)}$$

Engineering Brief about the elastic constants :

We have introduced a total of four elastic constants i.e. E, G, K and u. It may be seen that not all of these are independent of the others. In fact given any two of them, the other two can be determined. Further, it may be noted that

$$E = 3K(1-2\gamma)$$

or

$$K = \frac{E}{(1-2\gamma)}$$

if $\gamma = 0.5$; $K = \infty$

$$\text{Also } \epsilon_v = \frac{(1-2\gamma)}{E}(\sigma_x + \sigma_y + \sigma_z)$$

$$= \frac{(1-2\gamma)}{E}.3\sigma \text{ (for hydrostatic state of stress i.e } \sigma_x = \sigma_y = \sigma_z = \sigma \text{)}$$

Hence if u = 0.5, the value of K becomes infinite, rather than a zero value of E and the volumetric strain is zero or in other words, the material becomes incompressible

Further, it may be noted that under condition of simple tension and simple shear, all real materials tend to experience displacements in the directions of the applied forces and Under hydrostatic loading they tend to increase in volume. In other words the value of the elastic constants E, G and K cannot be negative

Therefore, the relations

E = 2 G (1 + u)

E = 3 K (1 - u)

Yields $\boxed{-1 \leq \nu \leq 0.5}$

In actual practice no real material has value of Poisson's ratio negative . Thus, the value of u cannot be greater than 0.5, if however u > 0.5 than $Î_v$ = -ve, which is physically

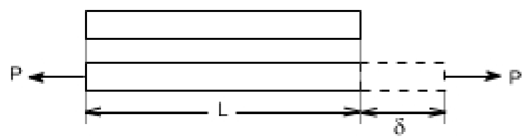
Unlikely because when the material is stretched its volume would always increase.

Members Subjected to Uniaxial Stress

Members in Uni – axial state of stress

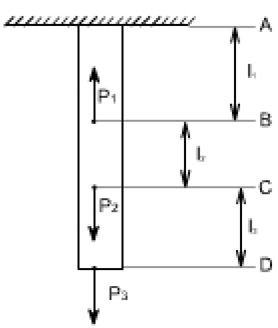
Introduction: [For members subjected to uniaxial state of stress]

For a prismatic bar loaded in tension by an axial force P, the elongation of the bar can be determined as



$$\delta = \frac{PL}{AE} \quad \text{——— (1)}$$

Suppose the bar is loaded at one or more intermediate positions, then equation (1) can be readily adapted to handle this situation, i.e. we can determine the axial force in each part of the bar i.e. parts AB, BC, CD, and calculate the elongation or shortening of each part separately, finally, these changes in lengths can be added algebraically to obtain the total charge in length of the entire bar.



When either the axial force or the cross – sectional area varies continuosly along the axis of the bar, then equation (1) is no longer suitable. Instead, the elongation can be found by considering a deferential element of a bar and then the equation (1) becomes

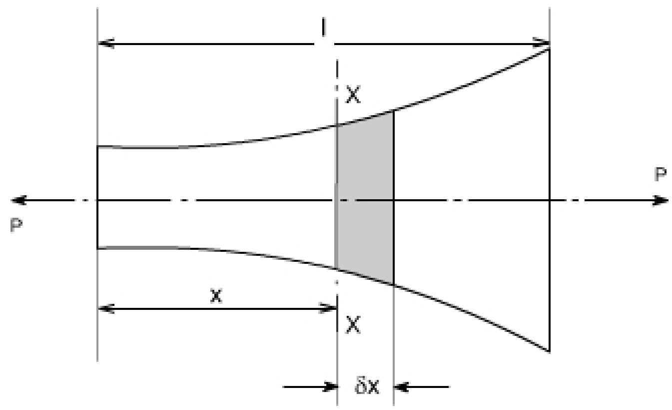
$$d\delta = \frac{P_x\,dx}{E.A_x}$$

$$\delta = \int_0^l \frac{P_x\,dx}{E.A_x}$$

i.e. the axial force $P_x$and area of the cross – section A_x must be expressed as functions of x. If the expressions for $P_x$and A_x are not too complicated, the integral can be evaluated analytically, otherwise Numerical methods or techniques can be used to evaluate these integrals.

stresses in Non – Uniform bars

Consider a bar of varying cross section subjected to a tensile force P as shown below.



Let

a = cross sectional area of the bar at a chosen section XX

then

Stress s = p / a

If E = Young's modulus of bar then the strain at the section XX can be calculated

Î = s / E

Then the extension of the short element d x. = Î .original length = s / E. d^x

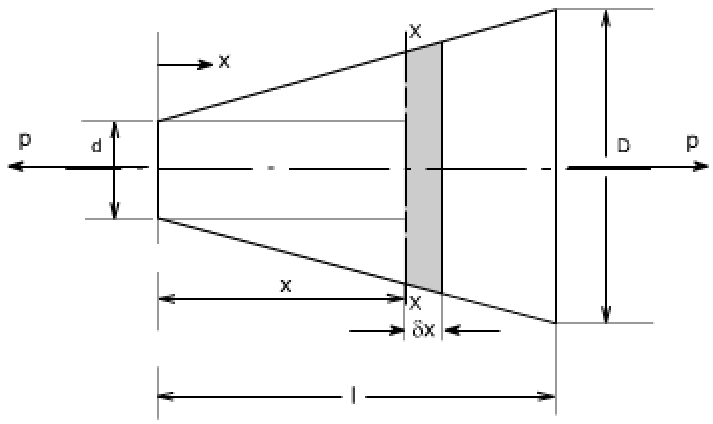
$$= \frac{P}{E.}\frac{\delta x}{a}$$

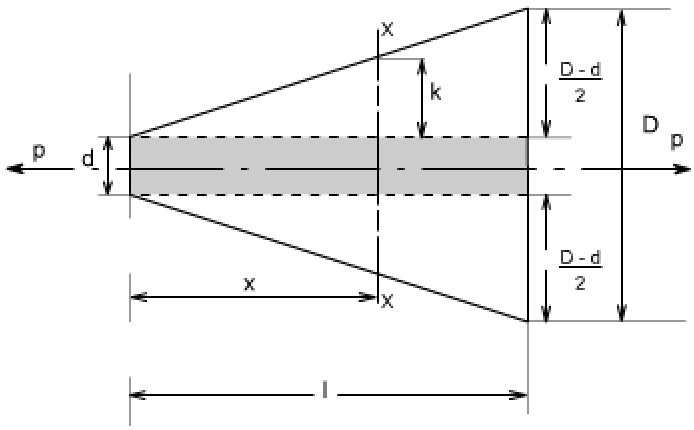
Thus,the extension for the entire bar is

$$\delta = \int_0^l \frac{P}{E}\frac{\delta x}{a}$$

$$\text{or total extension} = \frac{P}{E}\int_0^l \frac{\delta x}{a}$$

Now let us for example take a case when the bar tapers uniformly from d at x = 0 to D at x = l





In order to compute the value of diameter of a bar at a chosen location let us determine the value of dimension k, from similar triangles

$$\frac{(D-d)/2}{l} = \frac{k}{x}$$

$$\text{Thus, } k = \frac{(D-d)x}{2l}$$

therefore, the diameter 'y' at the X-section is

or = d + 2k

$$y = d + \frac{(D-d)x}{l}$$

Hence the cross –section area at section X- X will be

$$A_x \text{ or } a = \frac{\pi}{4} y^2$$
$$= \frac{\pi}{4}\left[d + (D - d)\frac{x}{l}\right]^2$$

hence the total extension of the bar will be given by expression

$$= \frac{P}{E}\int_0^l \frac{\delta x}{a}$$

substitituting the value of 'a' to get the total extention of the bar

$$= \frac{\pi P}{4E}\int_0^l \frac{\delta x}{\left[d + (D - d)\frac{x}{l}\right]^2}$$

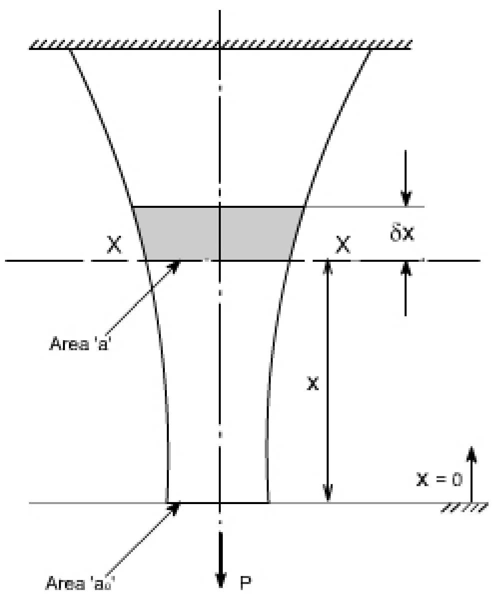
after carrying out the intergration we get

$$= -\frac{4.P.l}{\pi E}\left[\frac{1}{D} - \frac{1}{d}\right]$$
$$= \frac{4.P.l}{\pi E D.d}$$

hence the total strain in the bar $= \frac{\mathbf{4.P.l}}{\boldsymbol{\pi}\mathbf{E\ D.d}}$

An interesting problem is to determine the shape of a bar which would have a uniform stress in it under the action of its own weight and a load P.

let us consider such a bar as shown in the figure below:



The weight of the bar being supported under section XX is

$$= \int_0^x \rho g a \, dx$$

where ρ is density of the bar.

thus the stress at XX is

$$\sigma = \frac{P + \int_0^x \rho g a \, dx}{a}$$

$$\text{or } \sigma.a = P + \int_0^x \rho.g.a \, dx$$

Differentiating the above equation with respect to x we get

$$\sigma.\frac{da}{dx} = \rho.g.a$$

$$\frac{da}{a} = \frac{\rho.g}{\sigma}.dx$$

intergrating the above equation we get

$$\int \frac{da}{a} = \int \frac{\rho.g}{\sigma} dx$$

$$\log_e{}^a = \frac{\rho.g.x}{\sigma} + \text{constant}$$

In order to determine the constant of integration

let us apply the boundary conditions

at $x = 0; a = a_0$

thus, constant $= \log_e{}^{a_0}$

or

$$\log_e{}^a = \frac{\rho.g.x}{\sigma} + \log_e{}^{a_0}$$

$$\log_e\left(\frac{a}{a_0}\right) = \frac{\rho.g.x}{\sigma}$$

$$\text{or } \boxed{e^{\frac{\rho.g.x}{\sigma}} = \frac{a}{a_0}}$$

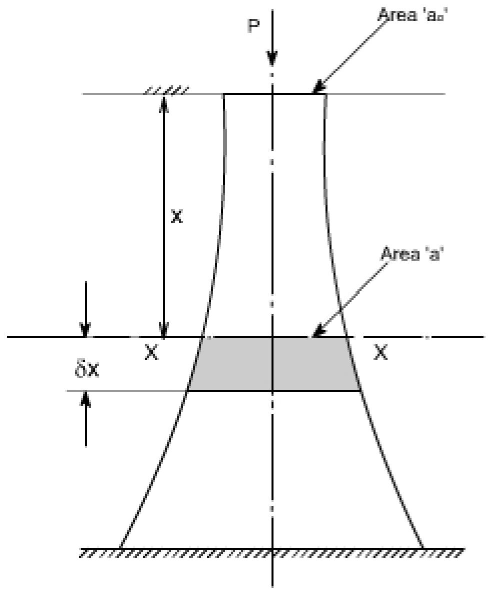
also at $x = 0$

$$\sigma = \frac{P}{a_0}$$

Thus,

$$\frac{a}{a_0} = e^{\frac{\rho.g.x.a_0}{P}}$$

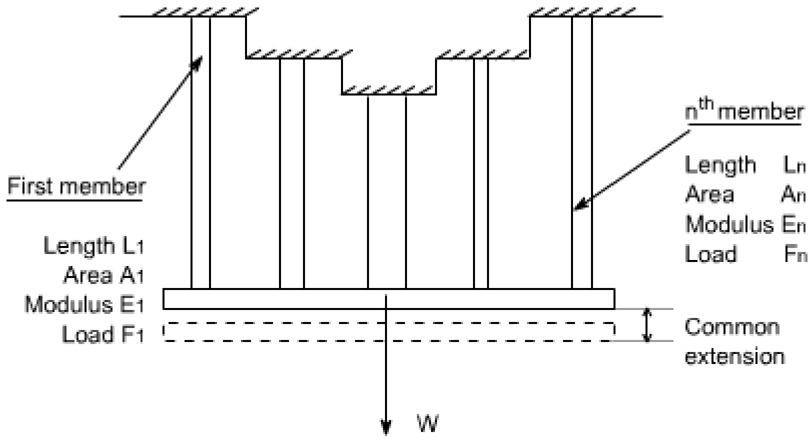
The same results are obtained if the bar is turned upside down and loaded as a column as shown in the figure below:



Thermal stresses, Bars subjected to tension and Compression

Compound bar: In certain application it is necessary to use a combination of elements or bars made from different materials, each material performing a different function. In over head electric cables or Transmission Lines for example it is often convenient to carry the current in a set of copper wires surrounding steel wires. The later being designed to support the weight of the cable over large spans. Such a combination of materials is generally termed compound bars.

Consider therefore, a compound bar consisting of n members, each having a different length and cross sectional area and each being of a different material. Let all member have a common extension ‘x' i.e. the load is positioned to produce the same extension in each member.



For the 'n' the members

$$\frac{\text{stress}}{\text{strain}} = E_n = \frac{F_n / A_n}{x_n / L_n}$$

$$= \frac{F_n \cdot L_n}{A_n \cdot x_n}$$

or $$F_n = \frac{E_n . A_n . x_n}{L_n} = \frac{E_n . A_n . x}{L_n} \quad .....(1)$$

Where F_n is the force in the nth member and A_n and L_n are its cross - sectional area and length.

Let W be the total load, the total load carried will be the sum of all loads for all the members.

$$W = \sum \frac{E_n . A_n . x}{L_n}$$

$$= x . \sum \frac{E_n . A_n}{L_n} \quad ....(2)$$

From equation (1), force in member 1 is given as

$$F_1 = \frac{E_1 . A_1 . x}{L_1}$$

from equation (2)

$$x = \frac{W}{\sum \frac{E_n . A_n}{L_n}}$$

$$\text{Thus, } F_1 = \frac{E_1 . A_1}{L_1} . \frac{W}{\sum \left(\frac{E_n . A_n}{L_n} \right)}$$

Therefore, each member carries a portion of the total load W proportional of EA / L value.

The above expression may be writen as $$F_1 = \frac{\frac{E_1.A_1}{L_1}}{\sum \frac{E_n.A_n}{L_n}}.W$$

if the length of each individual member in same then, we may write $$F_1 = \frac{E_1.A_1}{\sum E.A}.W$$

Thus, the stress in member '1' may be determined as $s_1 = F_1 / A_1$

Determination of common extension of compound bars: In order to determine the common extension of a compound bar it is convenient to consider it as a single bar of an imaginary material with an equivalent or combined modulus E_c.

Assumption: Here it is necessary to assume that both the extension and original lengths of the individual members of the compound bar are the same, the strains in all members will than be equal.

Total load on compound bar = $F_1 + F_2 + F_3 + \ldots\ldots\ldots + F_n$

where F_1 , F_2 ,….,etc are the loads in members 1,2 etc

But force = stress . area,therefore

$s\,(A_1 + A_2 + \ldots\ldots + A_n) = s_1 A_1 + s_2 A_2 + \ldots\ldots\ldots + s_n A_n$

Where s is the stress in the equivalent single bar

Dividing throughout by the common strain Î .

$$\frac{\sigma}{\in}(A_1 + A_2 + \ldots + A_n) = \frac{\sigma_1}{\in}A_1 + \frac{\sigma_2}{\in}A_2 + \ldots\ldots\frac{\sigma_n}{\in}A_n$$

$$\text{i.e } E_c(A_1 + A_2 + \ldots + A_n) = E_1A_1 + E_2A_2 + \ldots E_n.A_n$$

$$\text{or } E_c = \frac{E_1A_1 + E_2A_2 + \ldots E_n.A_n}{A_1 + A_2 + \ldots + A_n}$$

$$\text{or } E_c = \frac{\sum EA}{\sum A}$$

with an external load W applied stress in the equivalent bar may be computed as

$$\text{stress} = \frac{W}{\sum A}$$

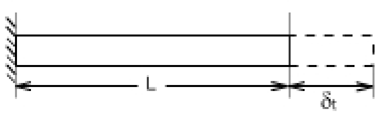
$$\text{strain inthe equivalent bar} = \frac{x}{L} = \frac{W}{\sum AE_c}$$

$$\text{hence commen extension } x = \frac{W.L}{E_c\sum A}$$

Compound bars subjected to temp. Change: Ordinary materials expand when heated and contract when cooled, hence, an increase in temperature produce a positive thermal strain. Thermal strains usually are reversible in a sense that the member returns to its original shape when the temperature return to its original value. However, there here are some materials which do not behave in this manner. These metals differ from ordinary materials in a sense that the strains are related non linearly to temperature and sometimes are irreversible .when a material is subjected to a change in temp. is a length will change by an amount.

d_t = a .L.t

or Î$_t$= a .L.t or s $_t$= E .a.t



a = coefficient of linear expansoin for the material

L = original Length

t = temp. change

Thus an increase in temperature produces an increase in length and a decrease in temperature results in a decrease in length except in very special cases of materials with zero or negative coefficients of expansion which need not to be considered here.

If however, the free expansion of the material is prevented by some external force, then a stress is set up in the material. They stress is equal in magnitude to that which would be produced in the bar by initially allowing the bar to its free length and then applying sufficient force to return the bar to its original length.

Change in Length = a L t

Therefore, strain = a L t / L

= a t

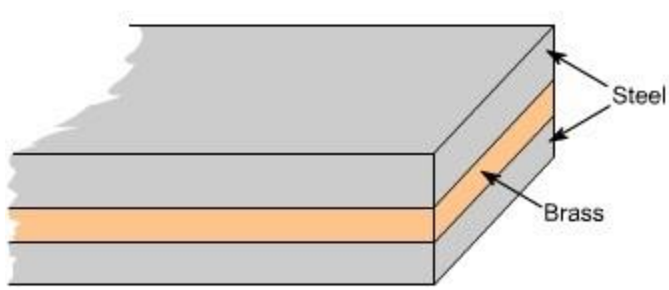
Therefore ,the stress generated in the material by the application of sufficient force to remove this strain

= strain x E

or Stress = E a t

Consider now a compound bar constructed from two different materials rigidly joined together, for simplicity.

Let us consider that the materials in this case are steel and brass.



If we have both applied stresses and a temp. change, thermal strains may be added to those given by generalized hook's law equation –e.g.

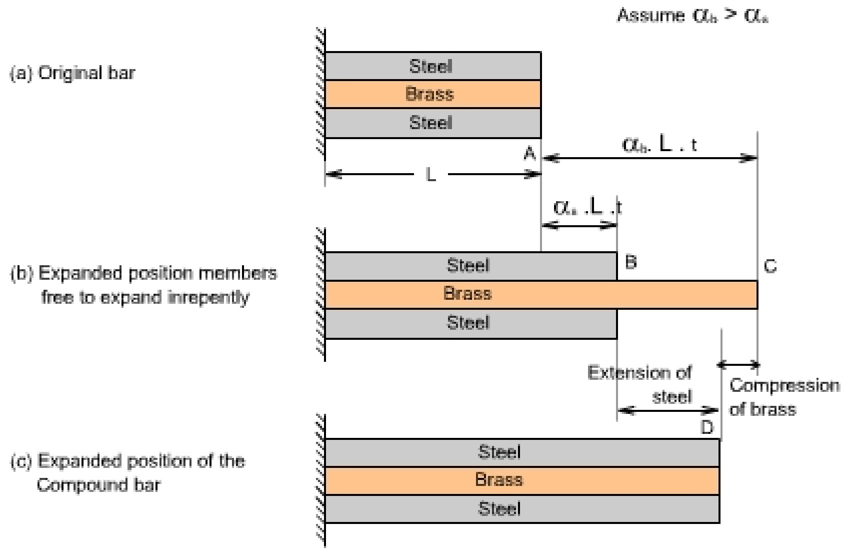
$$\in_x = \frac{1}{E}\left[\sigma_x - \gamma(\sigma_y + \sigma_z)\right] + \alpha\Delta t$$

$$\in_x = \frac{1}{E}\left[\sigma_y - \gamma(\sigma_x + \sigma_z)\right] + \alpha\Delta t$$

$$\in_x = \frac{1}{E}\left[\sigma_z - \gamma(\sigma_x + \sigma_y)\right] + \alpha\Delta t$$

While the normal strains a body are affected by changes in temperatures, shear strains are not. Because if the temp. of any block or element changes, then its size changes not its shape therefore shear strains do not change.

In general, the coefficients of expansion of the two materials forming the compound bar will be different so that as the temp. rises each material will attempt to expand by different amounts. Figure below shows the positions to which the individual materials will expand if they are completely free to expand (i.e not joined rigidly together as a compound bar). The extension of any Length L is given by a L t



In general, changes in lengths due to thermal strains may be calculated form equation d_t = a Lt, provided that the members are able to expand or contract freely, a situation that exists in statically determinates structures. As a consequence no stresses are generated in a statically determinate structure when one or more members undergo a uniform temperature change. If in a structure (or a compound bar), the free expansion or contraction is not allowed then the member becomes s statically indeterminate, which is just being discussed as an example of the compound bar and thermal stresses would be generated.

Thus the difference of free expansion lengths or so called free lengths

$= a_B.L.\ t - a_s\ .L\ .t$

$= (\ a_B - a_s\ ).L\ .t$

Since in this case the coefficient of expansion of the brass a_B is greater then that for the steel a_s. the initial lengths L of the two materials are assumed equal.

If the two materials are now rigidly joined as a compound bar and subjected to the same temp. rise, each materials will attempt to expand to its free length position but each will be affected by the movement of the other. The higher coefficient of expansion material (brass) will therefore, seek to pull the steel up to its free length position and conversely, the lower coefficient of expansion martial (steel) will try to hold the brass back. In practice a compromised is reached, the compound bar extending to the position shown in fig (c), resulting in an effective compression of the brass from its free length position and an effective extension of steel from its free length position.

Therefore, from the diagrams,we may conclude thefollowing

Conclusion 1.

Extension of steel + compression brass = difference in " free" length

Applying Newton 's law of equal action and reaction the following second Conclusion also holds good.

Conclusion 2.

The tensile force applied to the short member by the long member is equal in magnitude to the compressive force applied to long member by the short member.

Thus in this case

Tensile force in steel = compressive force in brass

These conclusions may be written in the form of mathematical equations as given below:

for conclusion 1

$$\frac{\sigma_s . L}{E_s} + \frac{\sigma_B . L}{E_B} = (\alpha_B - \alpha_s) L . t$$

for conclusion 2

$$\sigma_s . A_s = \sigma_B . A_B$$

Using these two equations, the magnitude of the stresses may be determined.

<u>**Energy Methods**</u>

Strain Energy

Strain Energy of the member is defined as the internal work done in defoming the body by the action of externally applied forces. This energy in elastic bodies is known as **elastic strain energy** :

Strain Energy in uniaxial Loading

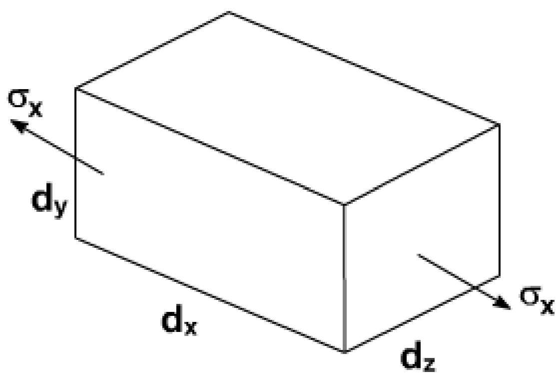


Fig .1

Let as consider an infinitesimal element of dimensions as shown in Fig .1. Let the element be subjected to normal stress s_x.

The forces acting on the face of this element is s_x. dy. dz

where

dydz = Area of the element due to the application of forces, the element deforms to an amount = $\hat{I}_x$ dx

$\hat{I}_x$ = strain in the material in x – direction

$$= \frac{\text{Change in length}}{\text{Orginal in length}}$$

Assuming the element material to be as linearly elastic the stress is directly proportional to strain as shown in Fig . 2.

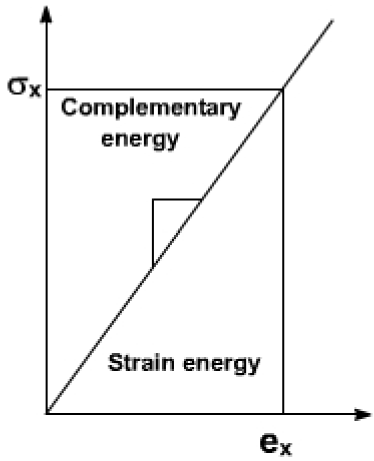


Fig .2

\ From Fig .2 the force that acts on the element increases linearly from zero until it attains its full value.

Hence average force on the element is equal to ½ s_x . dy. dz.

\ Therefore the workdone by the above force

Force = average force x deformed length

= ½ s_x. dydz . $\hat{I}_x$. dx

For a perfectly elastic body the above work done is the internal strain energy "du".

$$\boxed{du = \frac{1}{2}\sigma_x dydz\, \epsilon_x\, dx} \quad .....(2)$$

$$= \frac{1}{2}\sigma_x \epsilon_x\, dxdydz$$

$$\boxed{du = \frac{1}{2}\sigma_x\, \epsilon_x\, dv} \quad ....(3)$$

where dv = dxdydz

= Volume of the element

By rearranging the above equation we can write

$$U_o = \boxed{\frac{du}{dv} = \frac{1}{2}\sigma_x\, \epsilon_x} \quad .....(4)$$

The equation (4) represents the strain energy in elastic body per unit volume of the material its strain energy – density 'u_o' .

From Hook's Law for elastic bodies, it may be recalled that

$$\boxed{\sigma = E\epsilon}$$

$$\boxed{U_o = \frac{du}{dv} = \frac{\sigma_x^2}{2E} = \frac{E\epsilon_x^2}{2}} \qquad ......(5)$$

$$\boxed{U = \int_{Vol} \frac{\sigma_x^2}{2E} dv} \qquad ....(6)$$

In the case of a rod of uniform cross – section subjected at its ends an equal and opposite forces of magnitude P as shown in the Fig .3.

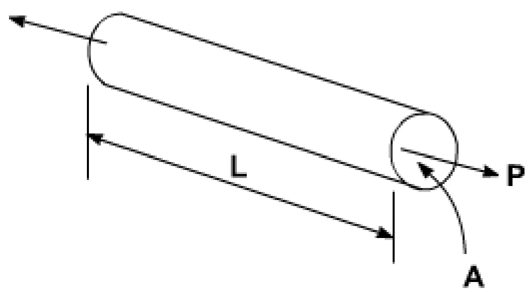


Fig .3

$$U = \int_{Vol}^{L} \frac{\sigma_x^2}{2E} dv \qquad \sigma_x = \frac{P}{A}$$

$$U = \int_0^L \frac{P^2}{2EA^2} Adx \qquad dv = Adx = \text{Element volume}$$

A = Area of the bar.

L = Length of the bar

$$\boxed{U = \frac{P^2L}{2AE}} \qquad .....(7)$$

Modulus of resilience :

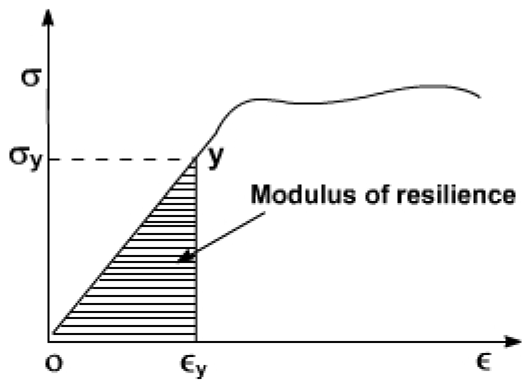


Fig .4

Suppose ‘ s_x‘ in strain energy equation is put equal to s_y i.e. the stress at proportional limit or yield point. The resulting strain energy gives an index of the materials ability to store or absorb energy without permanent deformation

So $$U_y = \frac{\sigma_y^2}{2E} \qquad .....(8)$$

The quantity resulting from the above equation is called the Modulus of resilience

The modulus of resilience is equal to the area under the straight line portion ‘OY' of the stress – strain diagram as shown in Fig .4 and represents the energy per unit volume that the material can absorb without yielding. Hence this is used to differentiate materials for applications where energy must be absorbed by members.

Modulus of Toughness :

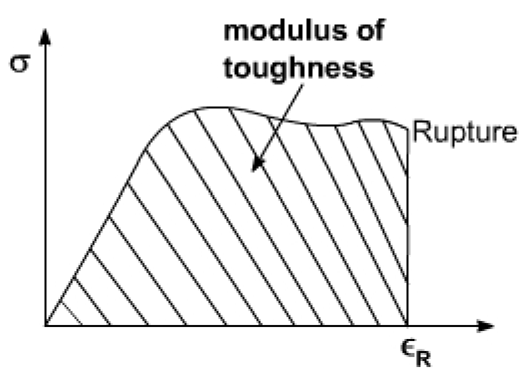


Fig .5

Suppose ‘Î' [strain] in strain energy expression is replaced by Î$_R$ strain at rupture, the resulting strain energy density is called modulus of toughness

$$U = \int_0^{\epsilon} E\,\epsilon_x\,dx = \frac{E\,\epsilon_R^2}{2}\,dv$$

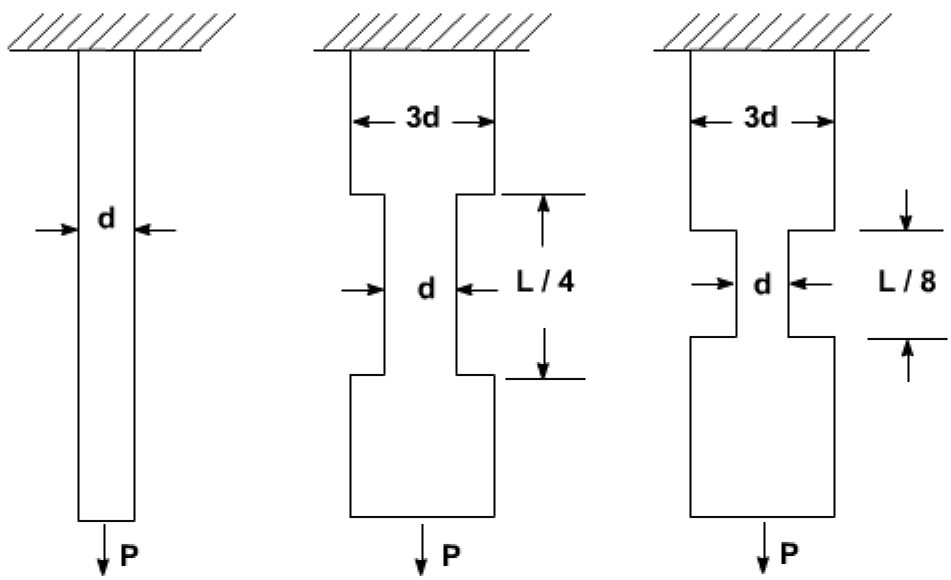
$$\boxed{U = \frac{E\,\epsilon_R^2}{2}} \qquad \ldots\ldots(9)$$

From the stress – strain diagram, the area under the complete curve gives the measure of modules of toughness. It is the materials.

Ability to absorb energy upto fracture. It is clear that the toughness of a material is related to its ductility as well as to its ultimate strength and that the capacity of a structure to withstand an impact Load depends upon the toughness of the material used.

ILLUSTRATIVE PROBLEMS

1. Three round bars having the same length 'L' but different shapes are shown in fig below. The first bar has a diameter 'd' over its entire length, the second had this diameter over one – fourth of its length, and the third has this diameter over one eighth of its length. All three bars are subjected to the same load P. Compare the amounts of strain energy stored in the bars, assuming the linear elastic behavior.



Solution :

1.The strain Energy of the first bar is expressed as

$$U_1 = \frac{P^2L}{2EA}$$

2.The strain Energy of the second bar is expressed as

$$U_2 = \frac{P^2(L/4)}{2EA} + \frac{P^2(3L/4)}{2E9A} = \frac{P^2L}{6EA}$$

$$\boxed{U_2 = \frac{U_1}{3}}$$

3.The strain Energy of the third bar is expressed as

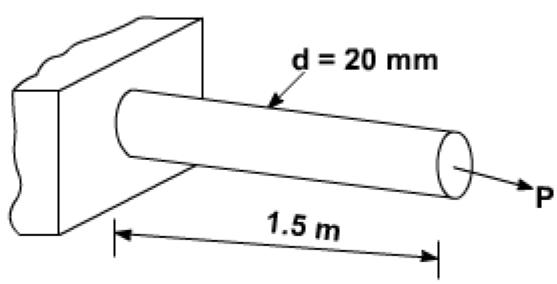
$$U_3 = \frac{P^2(L/8)}{2EA} + \frac{P^2(7L/8)}{2E(9A)}$$

$$U_3 = \frac{P^2L}{9EA}$$

$$\boxed{U_3 = \frac{2U_1}{9}}$$

From the above results it may be observed that the strain energy decreases as the volume of the bar increases.

2. Suppose a rod AB must acquire an elastic strain energy of 13.6 N.m using E = 200 GPa. Determine the required yield strength of steel. If the factor of safety w.r.t. permanent deformation is equal to 5.



Solution :

Factor of safety = 5

Therefore, the strain energy of the rod should be u = 5 [13.6] = 68 N.m

Strain Energy density

The volume of the rod is

$$V = AL = \frac{\pi}{4} d^2 L$$

$$= \frac{\pi}{4} 20 \times 1.5 \times 10^3$$

$$= 471 \times 10^3 \ mm^3$$

Yield Strength :

As we know that the modulus of resilience is equal to the strain energy density when maximum stress is equal to s_x .

$$U = \frac{\sigma_y^2}{2E}$$

$$0.144 = \frac{\sigma_y^2}{2 \times (200 \times 10^3)}$$

$$\boxed{\sigma_y = 200 \text{ Mpa}}$$

It is important to note that, since energy loads are not linearly related to the stress they produce, factor of safety associated with energy loads should be applied to the energy loads and not to the stresses.

Strain Energy in Bending :

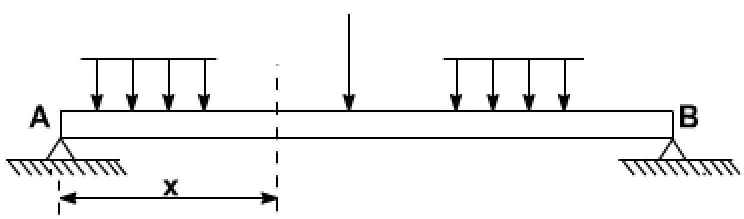


Fig .6

Consider a beam AB subjected to a given loading as shown in figure.

Let

M = The value of bending Moment at a distance x from end A.

From the simple bending theory, the normal stress due to bending alone is expressed as.

$$\sigma = \frac{MY}{I}$$

Substituting the above relation in the expression of strain energy

i.e. $U = \int \frac{\sigma^2}{2E} dv$

$$= \int \frac{M^2 . y^2}{2EI^2} dv \qquad .....(10)$$

Substituting dv = dxdA

Where dA = elemental cross-sectional area

$\frac{M^2 . y^2}{2EI^2} \rightarrow$ is a function of x alone

Now substitiuting for dy in the expression of U.

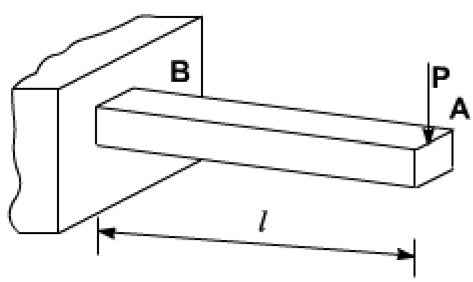
$$U = \int_0^L \frac{M^2}{2EI^2} \left(\int y^2 dA \right) dx \qquad .....(11)$$

We know $\int y^2 dA$ represents the moment of inertia 'I' of the cross-section about its neutral axis.

$$U = \int_0^L \frac{M^2}{2EI} dx \qquad .....(12)$$

ILLUSTRATIVE PROBLEMS

1. Determine the strain energy of a prismatic cantilever beam as shown in the figure by taking into account only the effect of the normal stresses.



Solution : The bending moment at a distance x from end A is defined as

$$M = -Px$$

Substituting the above value of M in the expression of strain energy we may write

$$U = \int_0^L \frac{P^2 x^2}{2EI} dx$$

$$\boxed{U = \int_0^L \frac{P^2 L^3}{EI}}$$

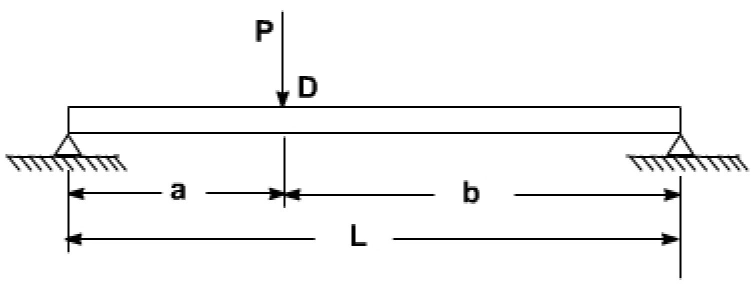
Problem 2 :

a. Determine the expression for strain energy of the prismatic beam AB for the loading as shown in figure below. Take into account only the effect of normal stresses due to bending.
b. Evaluate the strain energy for the following values of the beam

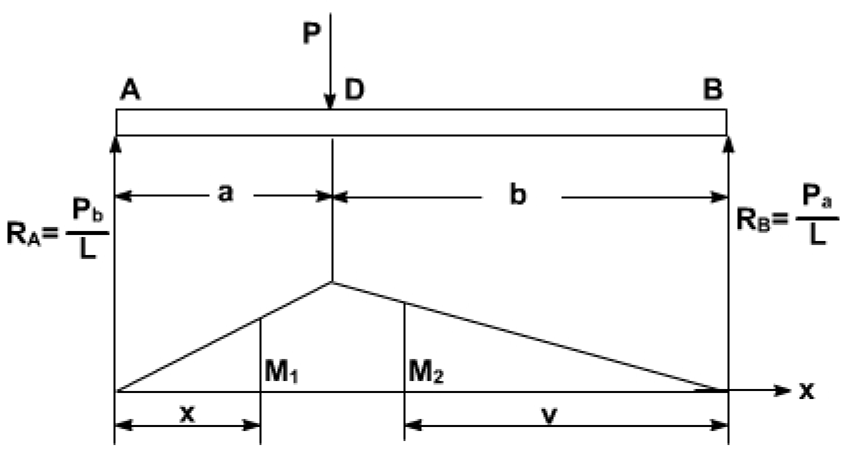
P = 208 KN ; L = 3.6 m = 3600 mm

A = 0.9 m = 90mm ; b = 2.7m = 2700 mm

E = 200 GPa ; I = 104 x 10^8 mm^4



Solution:



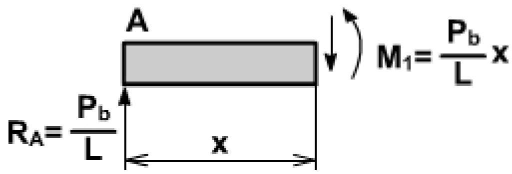
a.

Bending Moment : Using the free – body diagram of the entire beam, we may determine the values of reactions as follows:

$R_A = P_b / L$ $R_B = P_a / L$

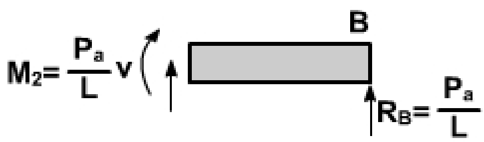
For Portion AD of the beam, the bending moment is

$$\boxed{M_1 = \frac{P_b}{L} x}$$



For Portion DB, the bending moment at a distance v from end B is

$$\boxed{M_2 = \frac{P_a}{L} x}$$



Strain Energy :

Since strain energy is a scalar quantity, we may add the strain energy of portion AD to that of DB to obtain the total strain energy of the beam.

$$U = U_{AD} + U_{DB}$$

$$= \int_0^a \frac{M_1^2}{2EI} dx + \int_0^b \frac{M_2^2}{2EI} dv$$

$$= \frac{1}{2EI} \int_0^a \left(\frac{P_b}{L} x \right)^2 dx + \frac{1}{2EI} \int_0^b \left(\frac{P_a}{L} v \right)^2 dx$$

$$= \frac{1}{2EI} \frac{P^2}{L^2} \left(\frac{b^2 a^3}{3} + \frac{a^2 b^3}{3} \right)$$

$$\boxed{U = \frac{P^2 a^2 b^2}{6EIL^2} (a + b)}$$

$$\text{Since}(a + b) = L$$

$$\boxed{U = \frac{P^2 a^2 b^2}{6EIL}}$$

b. Substituting the values of P, a, b, E, I, and L in the expression above.

$$U = \frac{(200 \times 10^3)^2 \times (900)^2 \times (2700)^2}{6\,(200 \times 10^3) \times (104 \times 10^6) \times (3600)} = 5.27 \times 10^7 \text{ KN.m}$$

Problem

3) Determine the modulus of resilience for each of the following materials.

a. Stainless steel . E = 190 GPa s_y = 260MPa

b. Malleable constantan E = 165GPa s_y = 230MPa

c. Titanium E = 115GPa s_y = 830MPa

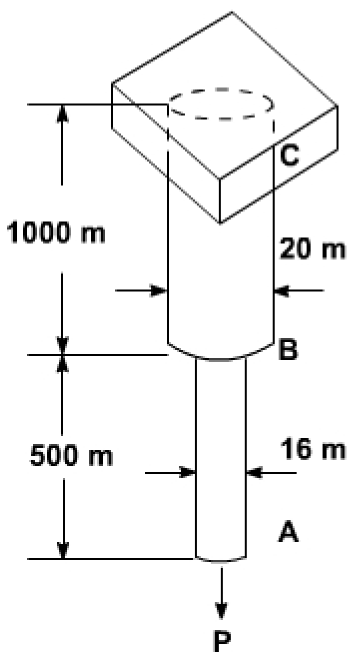
d. Magnesium E = 45GPa s_y = 200MPa

4) For the given Loading arrangement on the rod ABC determine

(a). The strain energy of the steel rod ABC when

P = 40 KN.

(b). The corresponding strain energy density in portions AB and BC of the rod.



UNIT 2

Members Subjected to Flexural Loads

Introduction:

In many engineering structures members are required to resist forces that are applied laterally or transversely to their axes. These type of members are termed as beam.

There are various ways to define the beams such as

<u>Definition I:</u> A beam is a laterally loaded member, whose cross-sectional dimensions are small as compared to its length.

<u>Definition II:</u> A beam is nothing simply a bar which is subjected to forces or couples that lie in a plane containing the longitudnal axis of the bar. The forces are understood to act perpendicular to the longitudnal axis of the bar.

<u>Definition III:</u> A bar working under bending is generally termed as a beam.

Materials for Beam:

The beams may be made from several usable engineering materials such commonly among them are as follows:

- Metal
- Wood
- Concrete
- Plastic

Examples of Beams:

Refer to the figures shown below that illustrates the beam

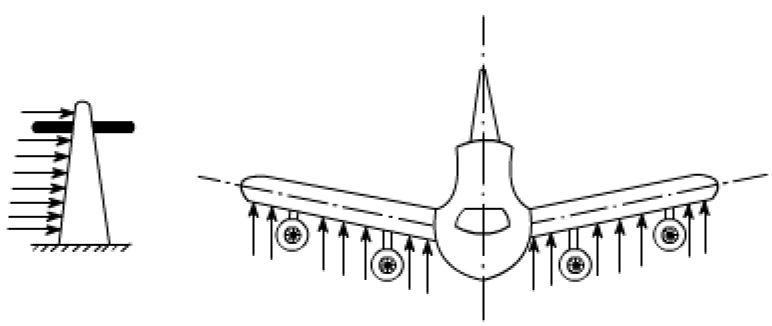


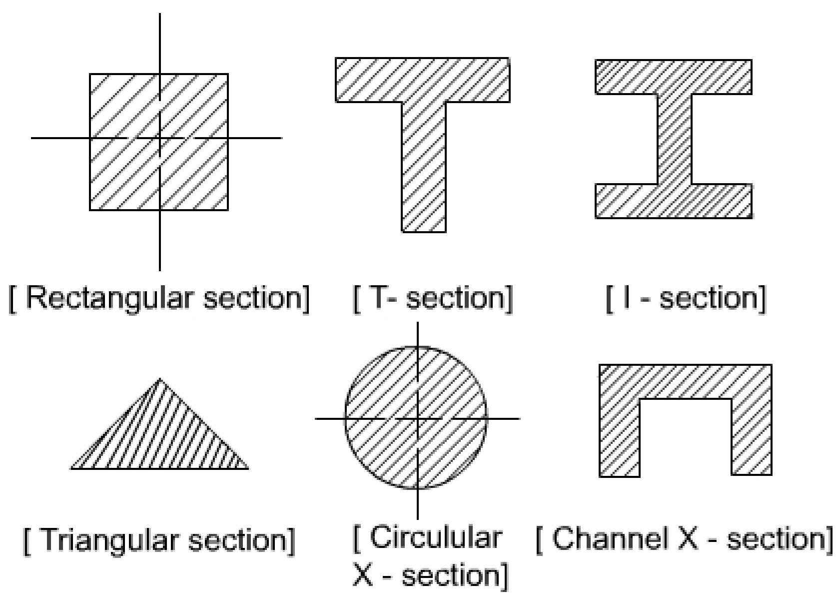
Fig 1 **Fig 2**

In the fig.1, an electric pole has been shown which is subject to forces occurring due to wind; hence it is an example of beam.

In the fig.2, the wings of an aeroplane may be regarded as a beam because here the aerodynamic action is responsible to provide lateral loading on the member.

Geometric forms of Beams:

The Area of X-section of the beam may take several forms some of them have been shown below:



Issues Regarding Beam:

Designer would be interested to know the answers to following issues while dealing with beams in practical engineering application

- At what load will it fail

- How much deflection occurs under the application of loads.

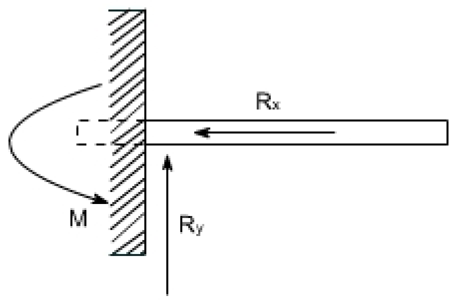
Classification of Beams:

Beams are classified on the basis of their geometry and the manner in which they are supported.

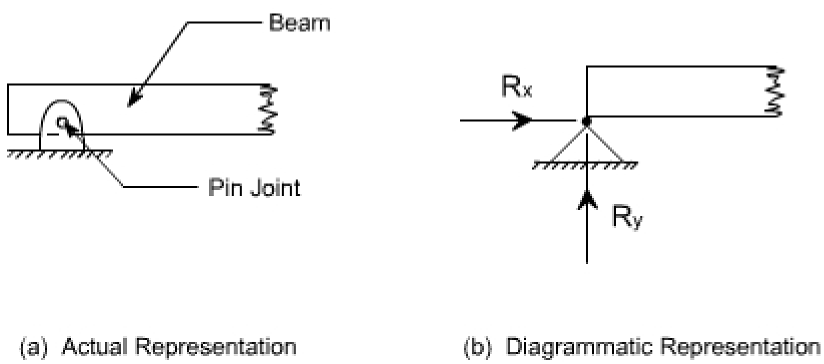
Classification I: The classification based on the basis of geometry normally includes features such as the shape of the X-section and whether the beam is straight or curved.

Classification II: Beams are classified into several groups, depending primarily on the kind of supports used. But it must be clearly understood why do we need supports. The supports are required to provide constrainment to the movement of the beams or simply the supports resists the movements either in particular direction or in rotational direction or both. As a consequence of this, the reaction comes into picture whereas to resist rotational movements the moment comes into picture. On the basis of the support, the beams may be classified as follows:

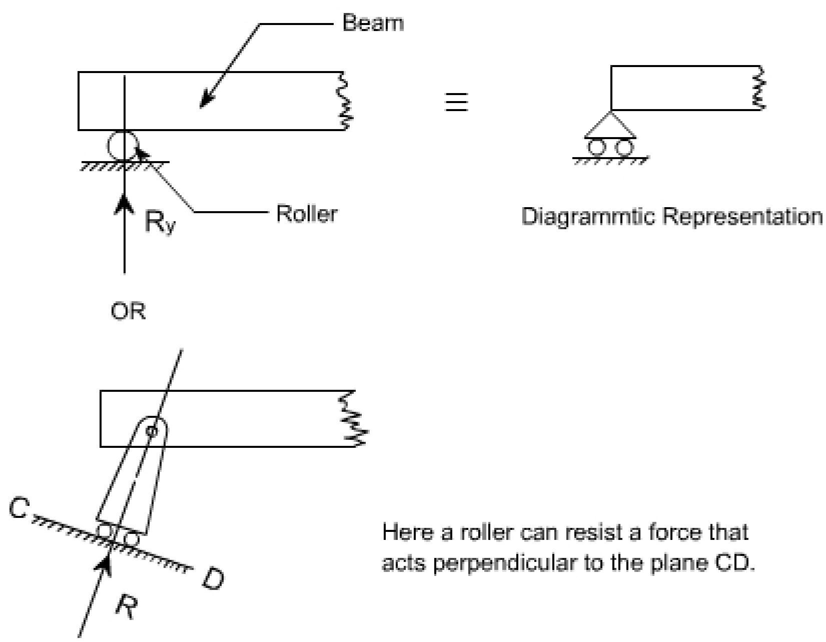
Cantilever Beam: A beam which is supported on the fixed support is termed as a cantilever beam: Now let us understand the meaning of a fixed support. Such a support is obtained by building a beam into a brick wall, casting it into concrete or welding the end of the beam. Such a support provides both the translational and rotational constrainment to the beam, therefore the reaction as well as the moments appears, as shown in the figure below



Simply Supported Beam: The beams are said to be simply supported if their supports creates only the translational constraints.



Some times the translational movement may be allowed in one direction with the help of rollers and can be represented like this



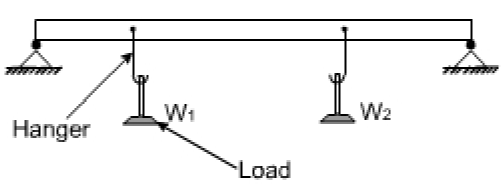
Statically Determinate or Statically Indeterminate Beams:

The beams can also be categorized as statically determinate or else it can be referred as statically indeterminate. If all the external forces and moments acting on it can be determined from the equilibrium conditions alone then. It would be referred as a statically determinate beam, whereas in the statically indeterminate beams one has to consider deformation i.e. deflections to solve the problem.

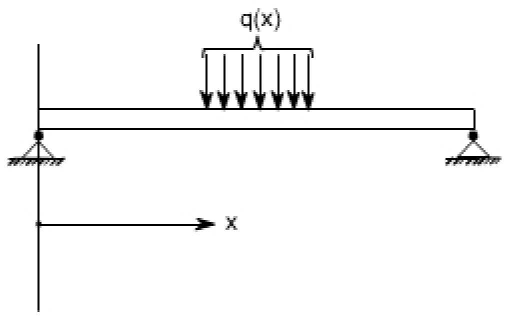
Types of loads acting on beams:

A beam is normally horizontal where as the external loads acting on the beams is generally in the vertical directions. In order to study the behaviors of beams under flexural loads. It becomes pertinent that one must be familiar with the various types of loads acting on the beams as well as their physical manifestations.

A. Concentrated Load: It is a kind of load which is considered to act at a point. By this we mean that the length of beam over which the force acts is so small in comparison to its total length that one can model the force as though applied at a point in two dimensional view of beam. Here in this case, force or load may be made to act on a beam by a hanger or though other means

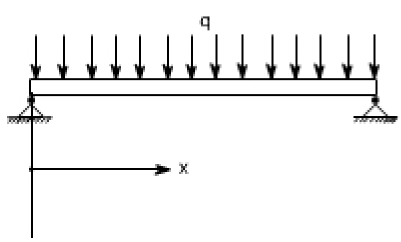


B. Distributed Load: The distributed load is a kind of load which is made to spread over a entire span of beam or over a particular portion of the beam in some specific manner

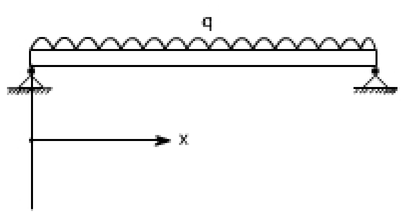


In the above figure, the rate of loading ‘q' is a function of x i.e. span of the beam, hence this is a non uniformly distributed load.

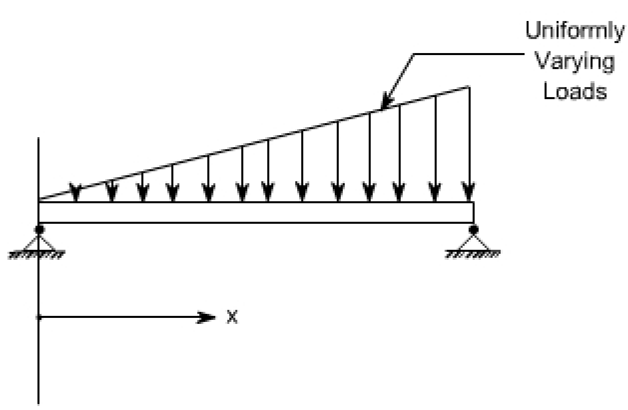
The rate of loading ‘q' over the length of the beam may be uniform over the entire span of beam, then we cell this as a uniformly distributed load (U.D.L). The U.D.L may be represented in either of the way on the beams



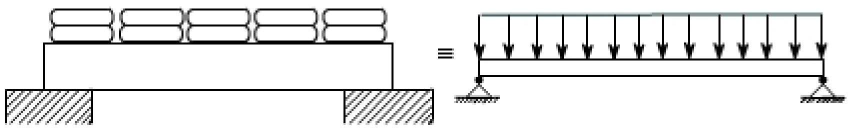
OR



some times the load acting on the beams may be the uniformly varying as in the case of dams or on inclind wall of a vessel containing liquid, then this may be represented on the beam as below:

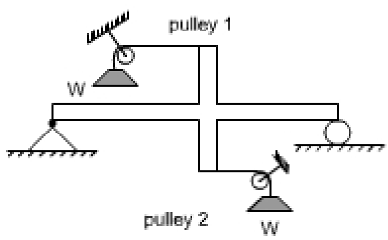


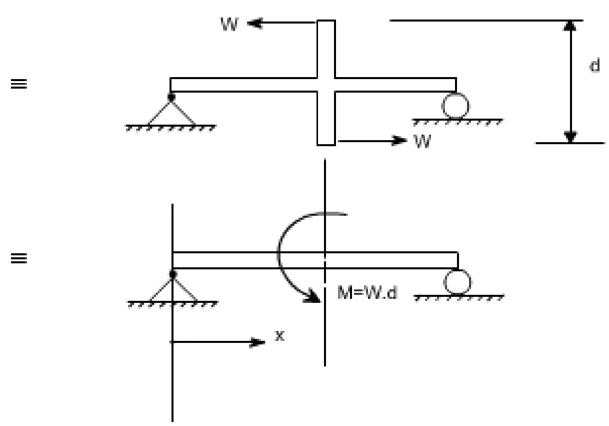
The U.D.L can be easily realized by making idealization of the ware house load, where the bags of grains are placed over a beam.



Concentrated Moment:

The beam may be subjected to a concentrated moment essentially at a point. One of the possible arrangement for applying the moment is being shown in the figure below:





Concept of Shear Force and Bending moment in beams:
When the beam is loaded in some arbitrarily manner, the internal forces and moments are developed and the terms shear force and bending moments come into pictures which are helpful to analyze the beams further. Let us define these terms

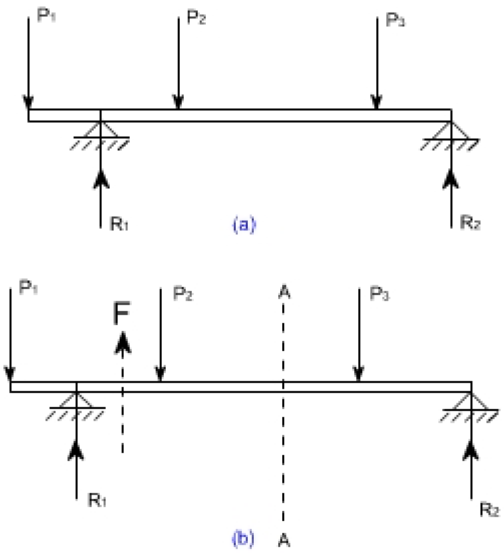


Fig 1

Now let us consider the beam as shown in fig 1(a) which is supporting the loads P_1, P_2, P_3 and is simply supported at two points creating the reactions R_1 and R_2 respectively. Now let us assume that the beam is to divided into or imagined to be cut into two portions at a section AA. Now let us assume that the resultant of loads and reactions to the left of AA is ‘F' vertically upwards, and since the entire beam is to remain in equilibrium, thus the resultant of forces to the right of AA must also be F, acting downwards. This forces ‘F' is as a shear force. The shearing force at any x-section of a beam represents the tendency for the portion of the beam to one side of the section to slide or shear laterally relative to the other portion.

Therefore, now we are in a position to define the shear force ‘F' to as follows:

At any x-section of a beam, the shear force ‘F' is the algebraic sum of all the lateral components of the forces acting on either side of the x-section.

Sign Convention for Shear Force:

The usual sign conventions to be followed for the shear forces have been illustrated in figures 2 and 3.

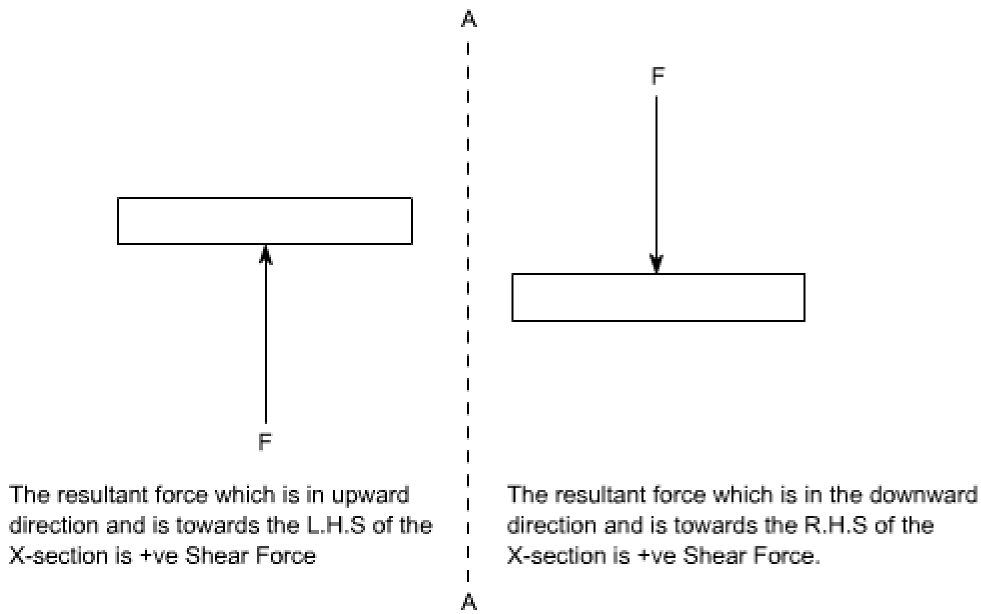


Fig 2: Positive Shear Force

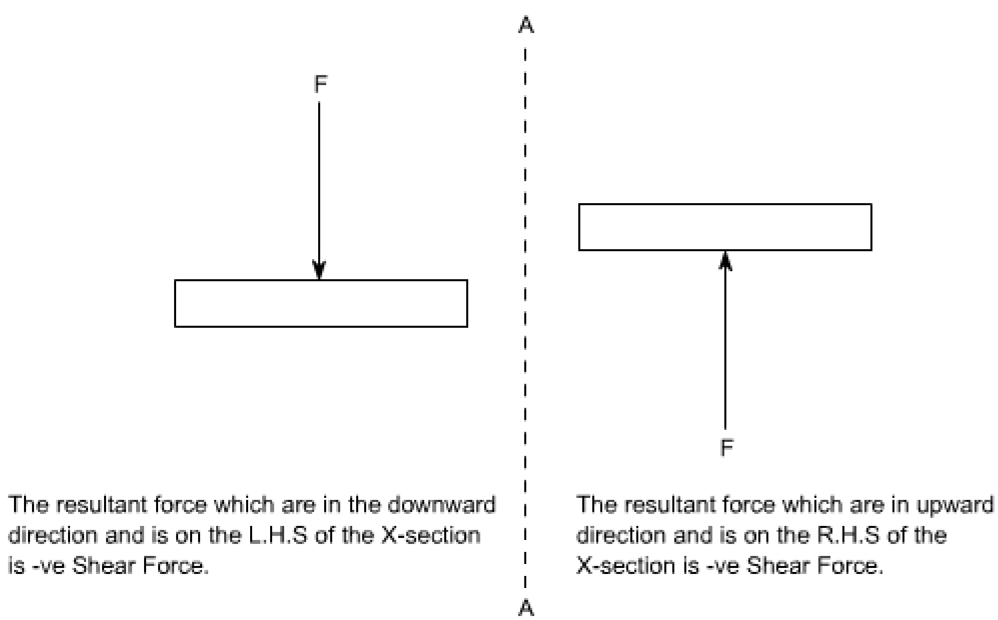


Fig 3: Negative Shear Force

Bending Moment:

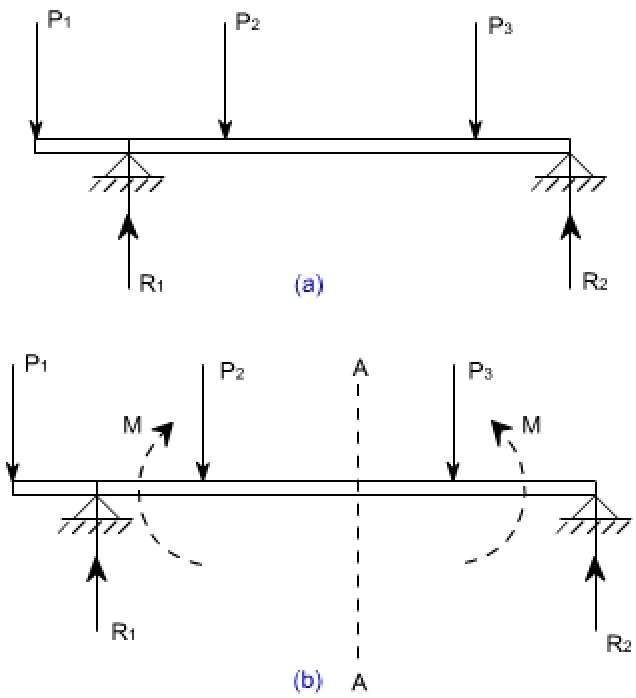


Fig 4

Let us again consider the beam which is simply supported at the two prints, carrying loads P_1, P_2 and P_3 and having the reactions R_1 and R_2 at the supports Fig 4. Now, let us imagine that the beam is cut into two potions at the x-section AA. In a similar manner, as done for the case of shear force, if we say that the resultant moment about the section AA of all the loads and reactions to the left of the x-section at AA is M in C.W direction, then moment of forces to the right of x-section AA must be 'M' in C.C.W. Then 'M' is called as the Bending moment and is abbreviated as B.M. Now one can define the bending moment to be simply as the algebraic sum of the moments about an x-section of all the forces acting on either side of the section

Sign Conventions for the Bending Moment:

For the bending moment, following sign conventions may be adopted as indicated in Fig 5 and Fig 6.

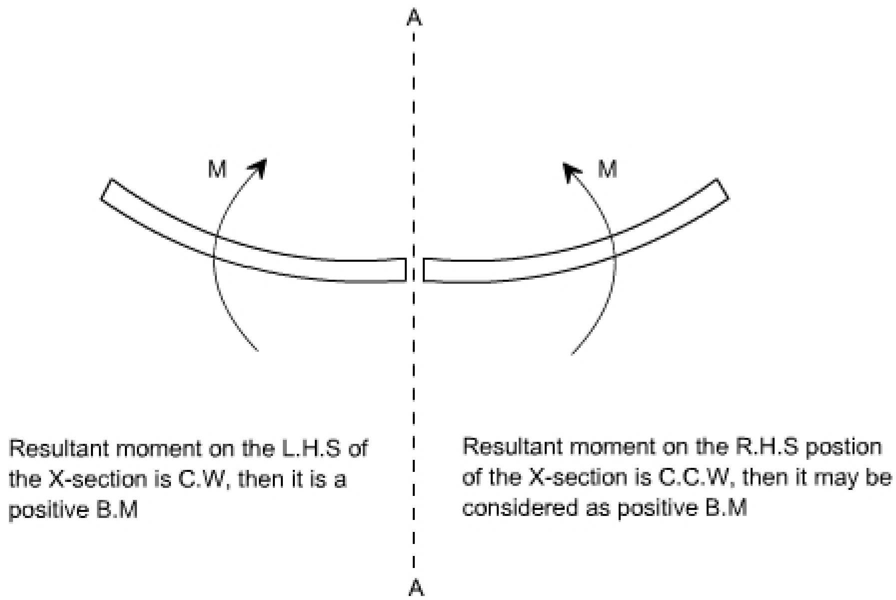


Fig 5: Positive Bending Moment

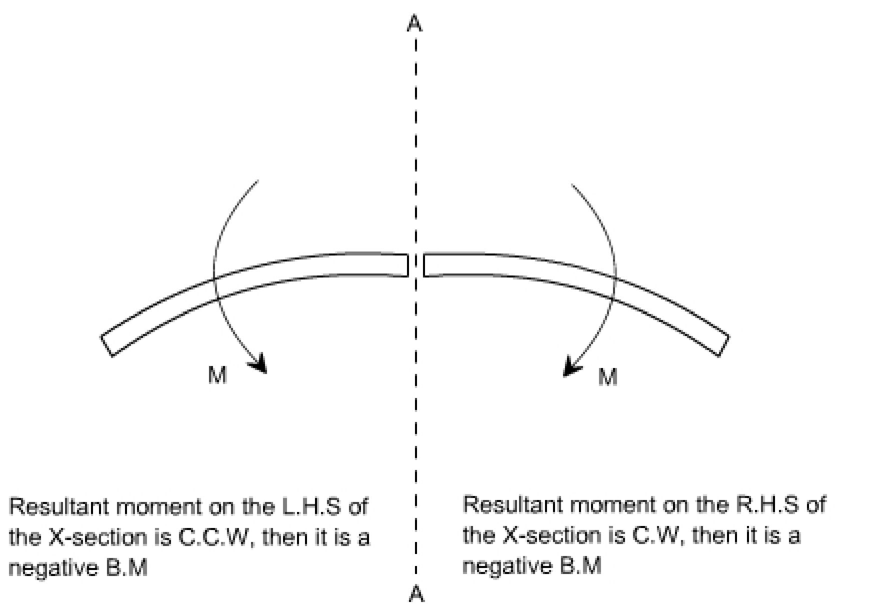


Fig 6: Negative Bending Moment

Some times, the terms 'Sagging' and Hogging are generally used for the positive and negative bending moments respectively.

Bending Moment and Shear Force Diagrams:

The diagrams which illustrate the variations in B.M and S.F values along the length of the beam for any fixed loading conditions would be helpful to analyze the beam further.

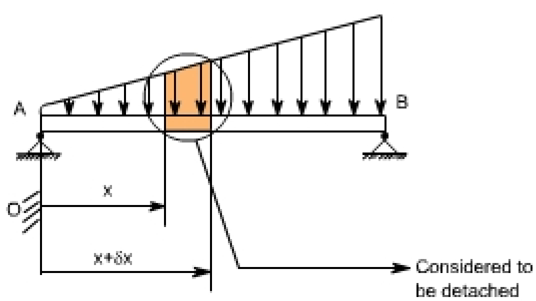
Thus, a shear force diagram is a graphical plot, which depicts how the internal shear force ‘F' varies along the length of beam. If x dentotes the length of the beam, then F is function x i.e. F(x).

Similarly a bending moment diagram is a graphical plot which depicts how the internal bending moment ‘M' varies along the length of the beam. Again M is a function x i.e. M(x).

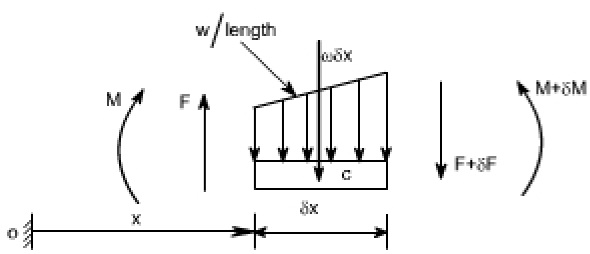
Basic Relationship Between The Rate of Loading, Shear Force and Bending Moment:

The construction of the shear force diagram and bending moment diagrams is greatly simplified if the relationship among load, shear force and bending moment is established.

Let us consider a simply supported beam AB carrying a uniformly distributed load w/length. Let us imagine to cut a short slice of length dx cut out from this loaded beam at distance ‘x' from the origin ‘0'.



Let us detach this portion of the beam and draw its free body diagram.



The forces acting on the free body diagram of the detached portion of this loaded beam are the following

- The shearing force F and F+ dF at the section x and x + dx respectively.

• The bending moment at the sections x and x + dx be M and M + dM respectively.

• Force due to external loading, if ‘w' is the mean rate of loading per unit length then the total loading on this slice of length dx is w. dx, which is approximately acting through the centre ‘c'. If the loading is assumed to be uniformly distributed then it would pass exactly through the centre ‘c'.

This small element must be in equilibrium under the action of these forces and couples.

Now let us take the moments at the point ‘c'. Such that

$$M + F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = M + \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + F.\frac{\delta x}{2} + \delta F.\frac{\delta x}{2} = \delta M \text{ [Neglecting the product of } \delta F \text{ and } \delta x \text{ being small quantities]}$$

$$\Rightarrow F.\delta x = \delta M$$

$$\Rightarrow F = \frac{\delta M}{\delta x}$$

Under the limits $\delta x \to 0$

$$\boxed{F = \frac{dM}{dx}} \qquad \ldots\ldots\ldots\ldots \qquad (1)$$

Resolving the forces vertically we get

$$w.\delta x + (F + \delta F) = F$$

$$\Rightarrow w = -\frac{\delta F}{\delta x}$$

Under the limits $\delta x \to 0$

$$\Rightarrow w = -\frac{dF}{dx} \text{ or } -\frac{d}{dx}\left(\frac{dM}{dx}\right)$$

$$\boxed{w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}} \qquad \ldots\ldots\ldots\ldots (2)$$

Conclusions: From the above relations,the following important conclusions may be drawn

• From Equation (1), the area of the shear force diagram between any two points, from the basic calculus is the bending moment diagram

$$M = \int F.dx$$

• The slope of bending moment diagram is the shear force,thus

$$F = \frac{dM}{dx}$$

Thus, if F=0; the slope of the bending moment diagram is zero and the bending moment is therefore constant.'

- The maximum or minimum Bending moment occurs where $\frac{dM}{dx} = 0.$

The slope of the shear force diagram is equal to the magnitude of the intensity of the distributed loading at any position along the beam. The –ve sign is as a consequence of our particular choice of sign conventions

Procedure for drawing shear force and bending moment diagram:

Preamble:

The advantage of plotting a variation of shear force F and bending moment M in a beam as a function of ‘x' measured from one end of the beam is that it becomes easier to determine the maximum absolute value of shear force and bending moment.

Further, the determination of value of M as a function of ‘x' becomes of paramount importance so as to determine the value of deflection of beam subjected to a given loading.

Construction of shear force and bending moment diagrams:

A shear force diagram can be constructed from the loading diagram of the beam. In order to draw this, first the reactions must be determined always. Then the vertical components of forces and reactions are successively summed from the left end of the beam to preserve the mathematical sign conventions adopted. The shear at a section is simply equal to the sum of all the vertical forces to the left of the section.

When the successive summation process is used, the shear force diagram should end up with the previously calculated shear (reaction at right end of the beam. No shear force acts through the beam just beyond the last vertical force or reaction. If the shear force diagram closes in this fashion, then it gives an important check on mathematical calculations.

The bending moment diagram is obtained by proceeding continuously along the length of beam from the left hand end and summing up the areas of shear force diagrams giving due regard to sign. The process of obtaining the moment diagram from the shear force diagram by summation is exactly the same as that for drawing shear force diagram from load diagram.

It may also be observed that a constant shear force produces a uniform change in the bending moment, resulting in straight line in the moment diagram. If no shear force exists along a certain portion of a beam, then it indicates that there is no change in moment takes place. It may also further observe that dm/dx= F therefore, from the fundamental theorem of calculus the maximum or minimum moment occurs where the shear is zero. In order to check the validity of the bending moment diagram, the terminal conditions for the moment must be satisfied. If the end is free or pinned, the computed sum must be equal to zero. If the end is built in, the moment computed by the summation must be equal to the one calculated initially for the reaction. These conditions must always be satisfied.

Illustrative problems:

In the following sections some illustrative problems have been discussed so as to illustrate the procedure for drawing the shear force and bending moment diagrams

1. A cantilever of length carries a concentrated load ‘W' at its free end.

Draw shear force and bending moment.

Solution:

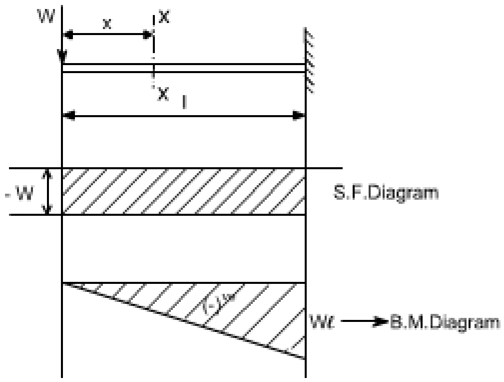
At a section a distance x from free end consider the forces to the left, then F = -W (for all values of x) -ve sign means the shear force to the left of the x-section are in downward direction and therefore negative

Taking moments about the section gives (obviously to the left of the section)

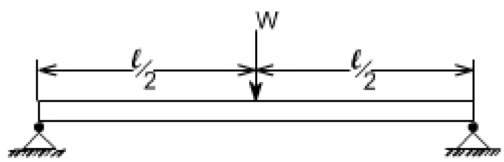
M = -Wx (-ve sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as –ve according to the sign convention)

so that the maximum bending moment occurs at the fixed end i.e. M = -W l

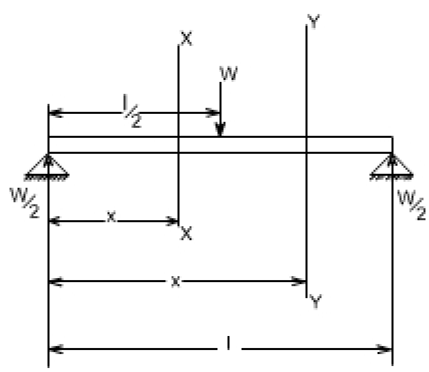
From equilibrium consideration, the fixing moment applied at the fixed end is Wl and the reaction is W. the shear force and bending moment are shown as,



2. Simply supported beam subjected to a central load (i.e. load acting at the mid-way)



By symmetry the reactions at the two supports would be W/2 and W/2. now consider any section X-X from the left end then, the beam is under the action of following forces.

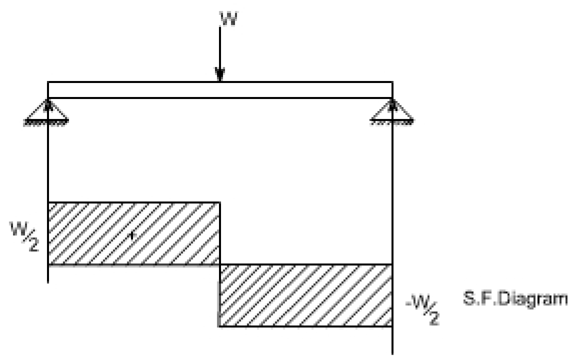


.So the shear force at any X-section would be = W/2 [Which is constant upto x < l/2]

If we consider another section Y-Y which is beyond l/2 then

$S.F_{Y\text{-}Y} = \frac{W}{2} - W = \frac{-W}{2}$ for all values greater = l/2

Hence S.F diagram can be plotted as,



.For B.M diagram:

If we just take the moments to the left of the cross-section,

$$B.M_{x\text{-}x} = \frac{W}{2}\,x \text{ for x lies between 0 and } l/2$$

$$B.M_{\text{at } x = \frac{l}{2}} = \frac{W}{2}\,\frac{l}{2} \text{ i.e B.M at x = 0}$$

$$= \frac{Wl}{4}$$

$$B.M_{Y\text{-}Y} = \frac{W}{2}\,x - W\left(x - \frac{l}{2}\right)$$

Again

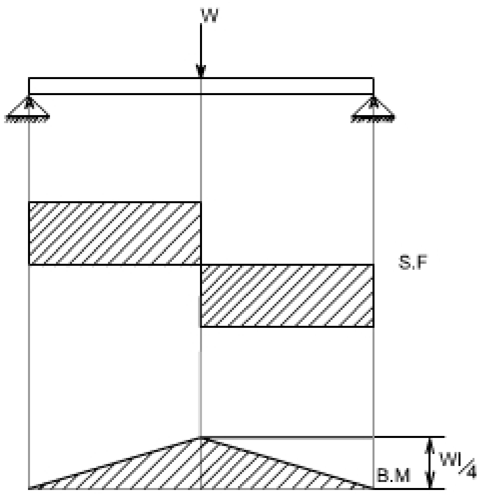
$$= \frac{W}{2}\,x - Wx + \frac{Wl}{2}$$

$$= -\frac{W}{2}\,x + \frac{Wl}{2}$$

$$B.M_{\text{at } x = l} = -\frac{Wl}{2} + \frac{Wl}{2}$$

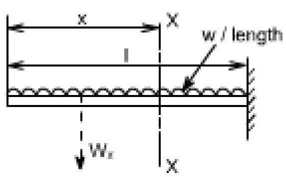
$$= 0$$

Which when plotted will give a straight relation i.e.



It may be observed that at the point of application of load there is an abrupt change in the shear force, at this point the B.M is maximum.

3. A cantilever beam subjected to U.d.L, draw S.F and B.M diagram.



Here the cantilever beam is subjected to a uniformly distributed load whose intensity is given w / length.

Consider any cross-section XX which is at a distance of x from the free end. If we just take the resultant of all the forces on the left of the X-section, then

$S.F_{xx}$ = -Wx for all values of ‘x'. ---------- **(1)**

$S.F_{xx}$ = 0

$S.F_{xx \text{ at } x=1}$ = -Wl

So if we just plot the equation No. (1), then it will give a straight line relation. Bending Moment at X-X is obtained by treating the load to the left of X-X as a concentrated load of the same value acting through the centre of gravity.

Therefore, the bending moment at any cross-section X-X is

$$B.M_{X\text{-}X} = -\,W\,x\,\frac{x}{2}$$

$$= -\,W\frac{x^2}{2}$$

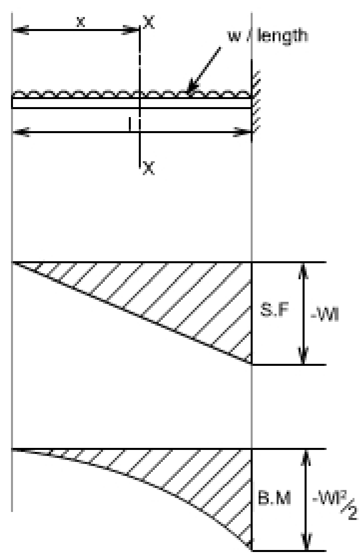
The above equation is a quadratic in x, when B.M is plotted against x this will produces a parabolic variation.

The extreme values of this would be at x = 0 and x = l

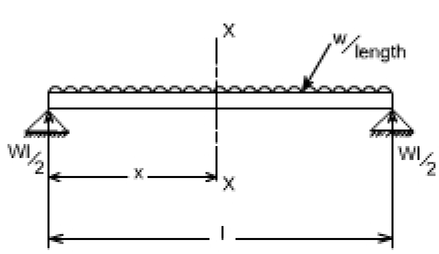
$$B.M_{at\ x\,=\,l} = -\frac{Wl^2}{2}$$

$$= \frac{Wl}{2} - Wx$$

Hence S.F and B.M diagram can be plotted as follows:



4. <u>Simply supported beam subjected to a uniformly distributed load [U.D.L].</u>



The total load carried by the span would be

= intensity of loading x length

= w x l

By symmetry the reactions at the end supports are each wl/2

If x is the distance of the section considered from the left hand end of the beam.

S.F at any X-section X-X is

$$= \frac{Wl}{2} - Wx$$
$$= W\left(\frac{l}{2} - x\right)$$

Giving a straight relation, having a slope equal to the rate of loading or intensity of the loading.

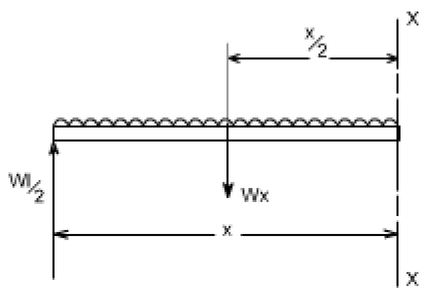
$$S.F_{at\ x=0} = \frac{wl}{2} - wx$$

so at

$S.F_{at\ x=\frac{l}{2}} = 0$ hence the S.F is zero at the centre

$$S.F_{at\ x=l} = -\frac{Wl}{2}$$

The bending moment at the section x is found by treating the distributed load as acting at its centre of gravity, which at a distance of x/2 from the section



$$B.M_{x\text{-}x} = \frac{Wl}{2}x - Wx.\frac{x}{2}$$

so the

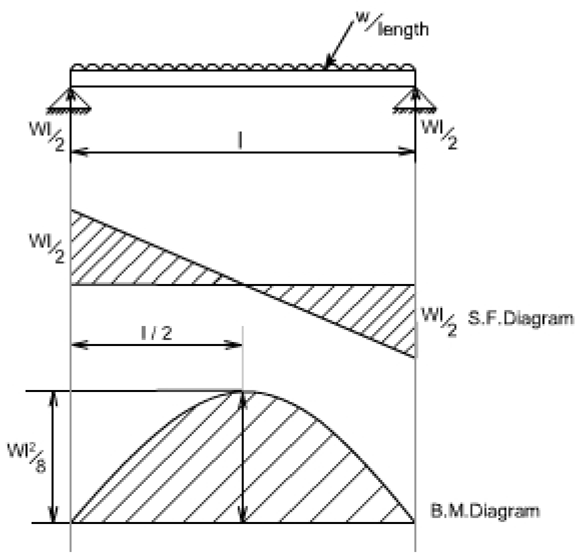
$$= W.\frac{x}{2}(l - 2) \quad \ldots\ldots..(2)$$

$$B.M_{at\ x\ =\ 0} = 0$$

$$B.M_{at\ x\ =\ l} = 0$$

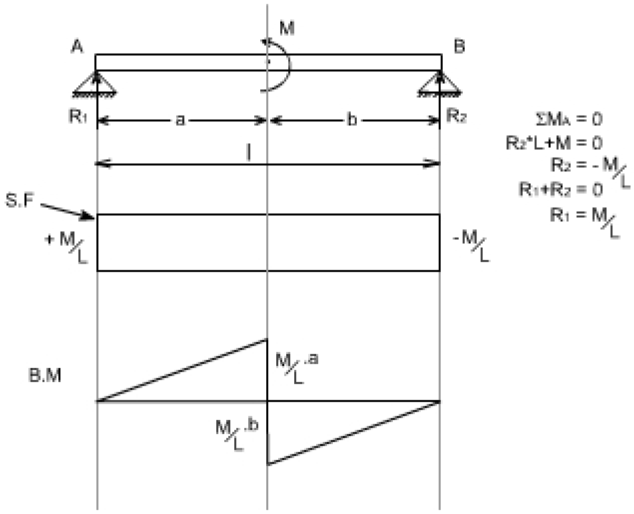
$$B.M\Big|_{at\ x\ =\ l} = -\frac{Wl^2}{8}$$

So the equation (2) when plotted against x gives rise to a parabolic curve and the shear force and bending moment can be drawn in the following way will appear as follows:



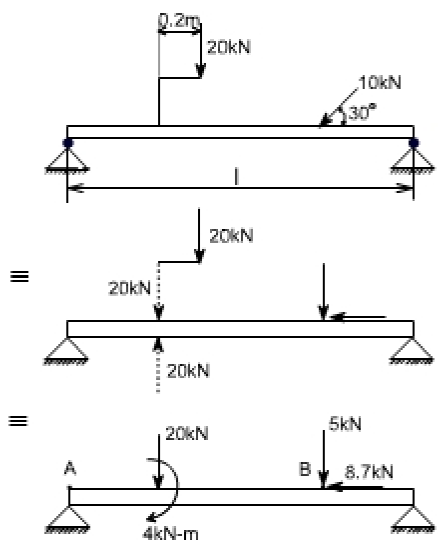
5. Couple.

When the beam is subjected to couple, the shear force and Bending moment diagrams may be drawn exactly in the same fashion as discussed earlier.



6. Eccentric loads.

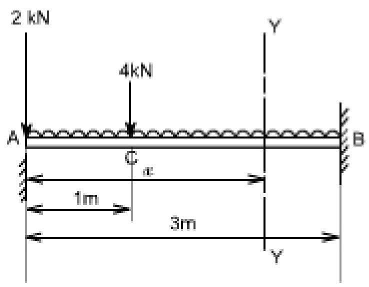
When the beam is subjected to an eccentric loads, the eccentric load are to be changed into a couple/ force as the case may be, In the illustrative example given below, the 20 kN load acting at a distance of 0.2m may be converted to an equivalent of 20 kN force and a couple of 2 kN.m. similarly a 10 kN force which is acting at an angle of 30^0 may be resolved into horizontal and vertical components.The rest of the procedure for drawing the shear force and Bending moment remains the same.



6. Loading changes or there is an abrupt change of loading:

When there is an aabrupt change of loading or loads changes, the problem may be tackled in a systematic way.consider a cantilever beam of 3 meters length. It carries a uniformly distributed load of 2 kN/m and a concentrated loads of 2kN at the free end and 4kN at 2 meters from fixed end.The shearing force and bending moment diagrams are required to be drawn and state the maximum values of the shearing force and bending moment.

Solution



Consider any cross section x-x, at a distance x from the free end

Shear Force at x-x = -2 -2x 0 < x < 1

S.F at x = 0 i.e. at A = -2 kN

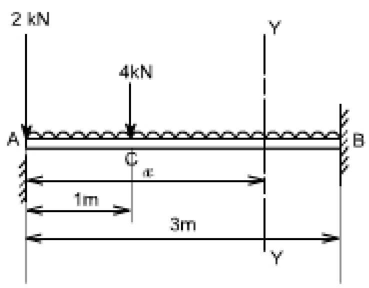
S.F at x = 1 = -2-2 = - 4kN

S.F at C (x = 1) = -2 -2x - 4 Concentrated load

= - 2 - 4 -2x1 kN

= - 8 kN

Again consider any cross-section YY, located at a distance x from the free end



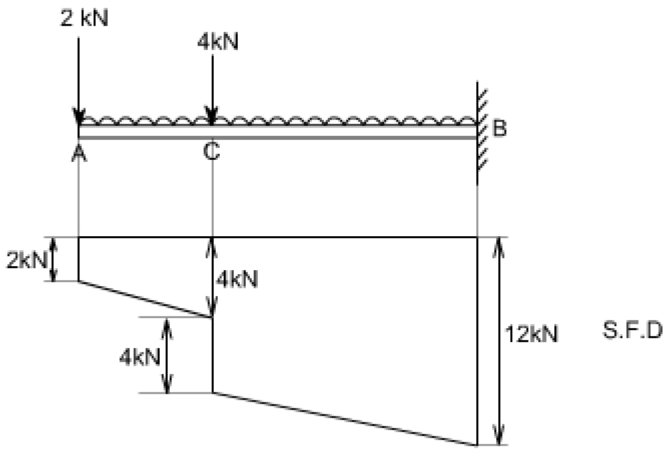
S.F at Y-Y = -2 - 2x - 4 1< x < 3

This equation again gives S.F at point C equal to -8kN

S.F at x = 3 m = -2 -4 -2x3

= -12 kN

Hence the shear force diagram can be drawn as below:



For bending moment diagrams – Again write down the equations for the respective cross sections, as consider above

Bending Moment at xx = -2x - 2x.x/2 valid upto AC

B.M at x = 0 = 0

B.M at x =1m = -3 kN.m

For the portion CB, the bending moment equation can be written for the x-section at Y-Y .

B.M at YY = -2x - 2x.x/2 - 4(x -1)

This equation again gives,

B.M at point C = - 2.1 - 1 - 0 i.e. at x = 1

= -3 kN.m

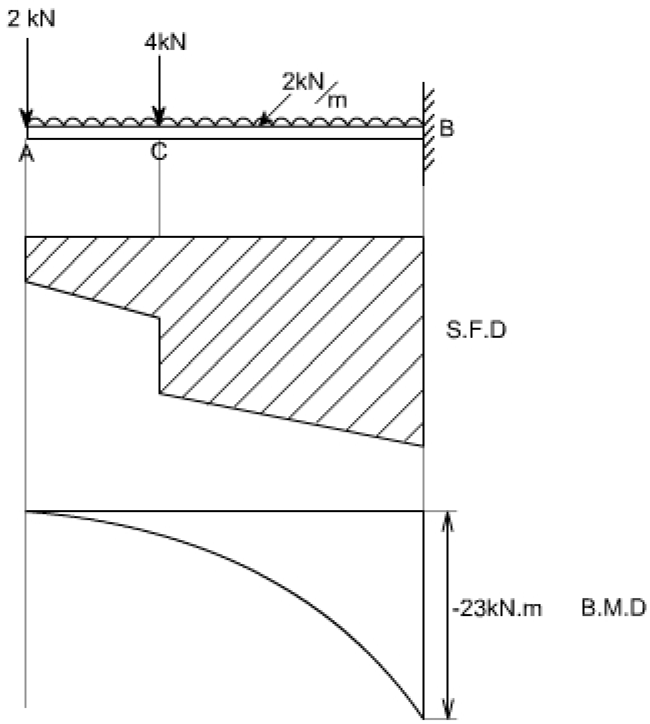
B.M at point B i.e. at x = 3 m

= - 6 - 9 - 8

= - 23 kN-m

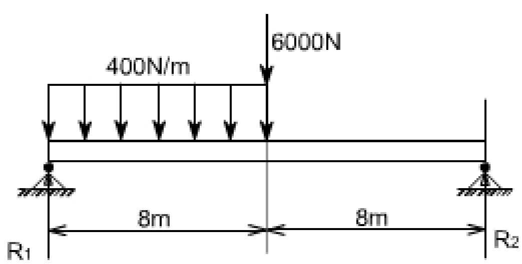
The variation of the bending moment diagrams would obviously be a parabolic curve

Hence the bending moment diagram would be



7. Illustrative Example :

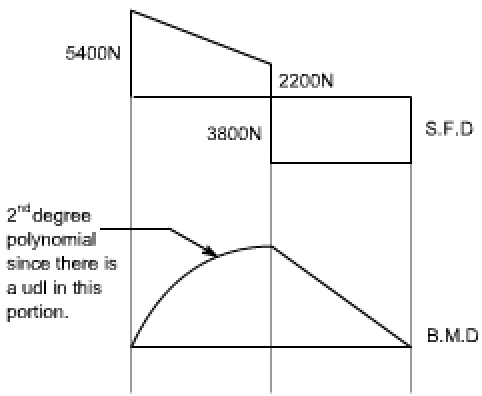
In this there is an abrupt change of loading beyond a certain point thus, we shall have to be careful at the jumps and the discontinuities.



For the given problem, the values of reactions can be determined as

R2 = 3800N and R1 = 5400N

The shear force and bending moment diagrams can be drawn by considering the X-sections at the suitable locations.



8. Illustrative Problem :

The simply supported beam shown below carries a vertical load that increases uniformly from zero at the one end to the maximum value of 6kN/m of length at the other end .Draw the shearing force and bending moment diagrams.

Solution

Determination of Reactions

For the purpose of determining the reactions R1 and R2 , the entire distributed load may be replaced by its resultant which will act through the centroid of the triangular loading diagram.

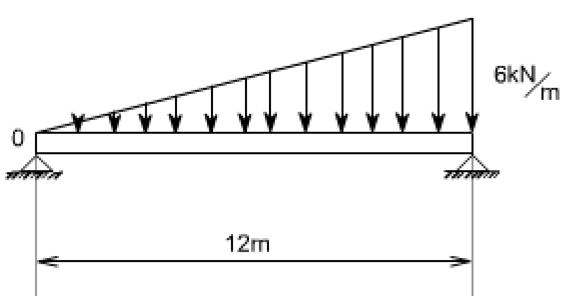
So the total resultant load can be found like this-

Average intensity of loading = (0 + 6)/2

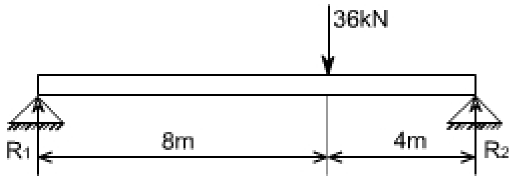
= 3 kN/m

Total Load = 3 x 12

= 36 kN



Since the centroid of the triangle is at a 2/3 distance from the one end, hence 2/3 x 3 = 8 m from the left end support.



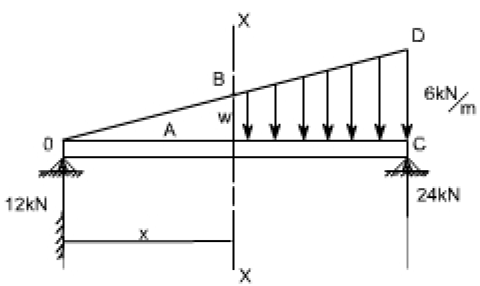
Now taking moments or applying conditions of equilibrium

36 x 8 = R2 x 12

R1 = 12 kN

R2 = 24 kN

Note: however, this resultant can not be used for the purpose of drawing the shear force and bending moment diagrams. We must consider the distributed load and determine the shear and moment at a section x from the left hand end.



Consider any X-section X-X at a distance x, as the intensity of loading at this X-section, is unknown let us find out the resultant load which is acting on the L.H.S of the X-section X-X, hence

So consider the similar triangles

OAB & OCD

$$\frac{w}{6} = \frac{x}{12}$$

$$w = \frac{x}{2} \quad k\frac{N}{m}$$

In order to find out the total resultant load on the left hand side of the X-section

Find the average load intensity

$$= \frac{0 + \frac{x}{2}}{2}$$

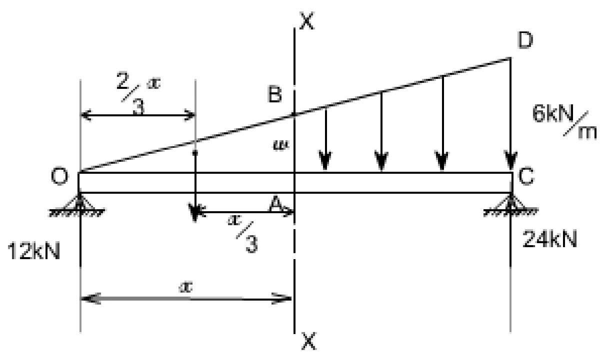
$$= \frac{x}{4}\, k\frac{N}{m}$$

Therefore the total load over the length x would be

$$= \frac{x}{4}.x \text{ kN}$$

$$= \frac{x^2}{4} \text{ kN}$$

Now these loads will act through the centroid of the triangle OAB. i.e. at a distance 2/3 x from the left hand end. Therefore, the shear force and bending momemt equations may be written as



$$S.F_{at\ xx} = \left(12 - \frac{x^2}{4}\right) \text{ kN}$$

valid for all values of x ....(1)

$$B.M_{at\ xx} = 12\,x - \frac{x^2}{4} \cdot \frac{x}{3}$$

$$B.M_{at\ xx} = 12\,x - \frac{x^3}{12} \text{ kN-m}$$

valid for all values of x ....(2)

$$S.F_{at\ x=0} = 12 \text{ kN}$$

$$S.F_{at\ x=12m} = 12 - \frac{12 \times 12}{4}$$

$$= -24 \text{ kN}$$

In order to find out the point where S.F is zero

$$\left(12 - \frac{x^2}{4}\right) = 0$$

x = 6.92 m (selecting the positive values)

Again

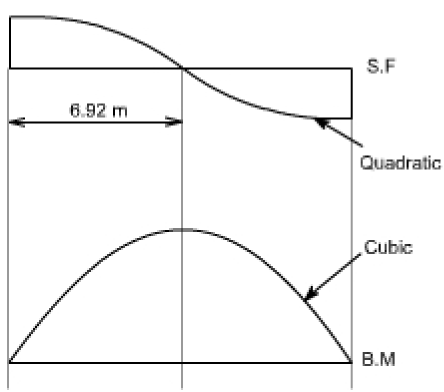
$$B.M_{at\ x=0} = 0$$

$$B.M_{at\ x=12} = 12 \times 12 - \frac{12^3}{12}$$

$$= 0$$

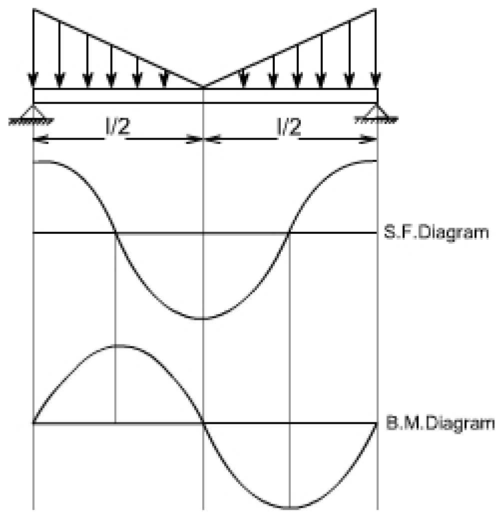
$$B.M_{at\ x=6.92} = 12 \times 6.92 - \frac{6.92^3}{12}$$

$$= 55.42 \text{ kN - m}$$



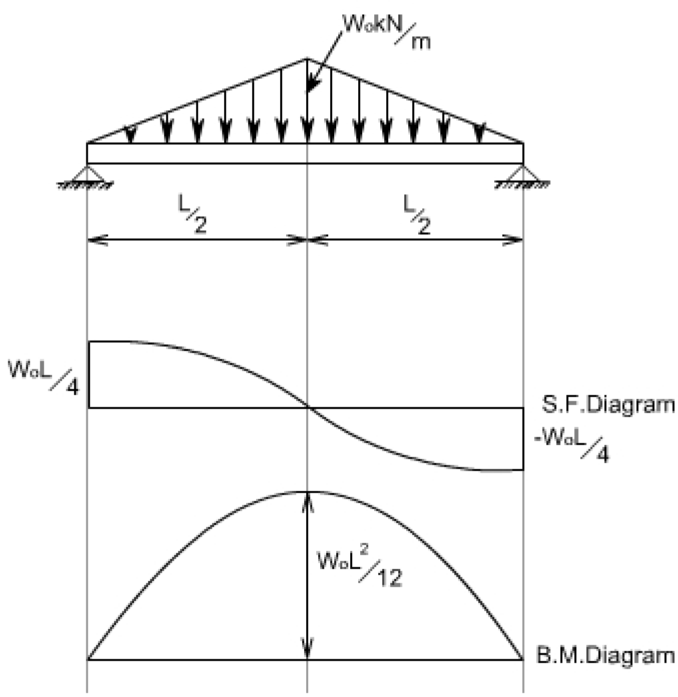
9. Illustrative problem :

In the same way, the shear force and bending moment diagrams may be attempted for the given problem



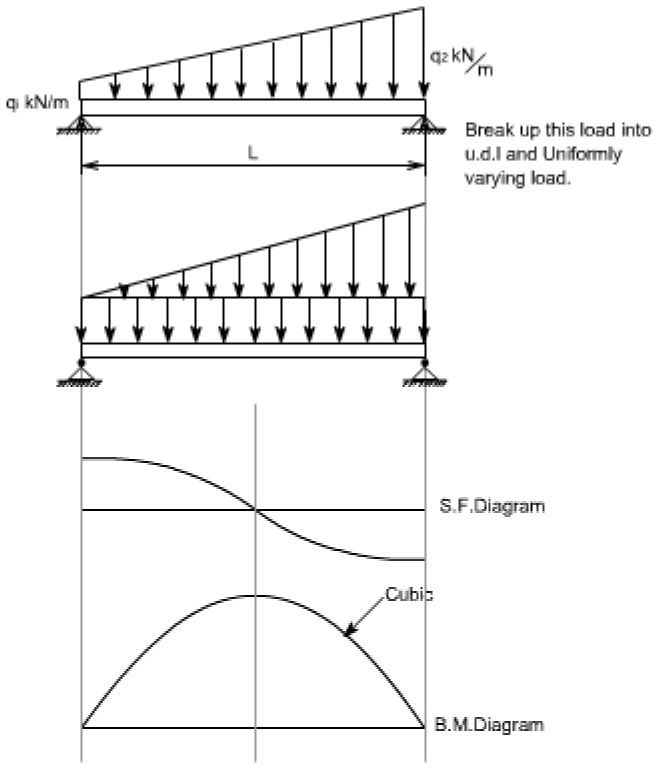
10. Illustrative problem :

For the uniformly varying loads, the problem may be framed in a variety of ways, observe the shear force and bending moment diagrams

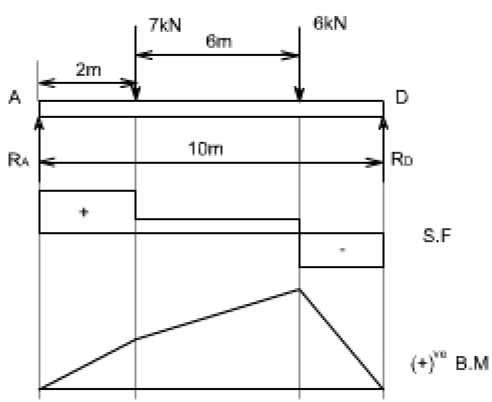


11. Illustrative problem :

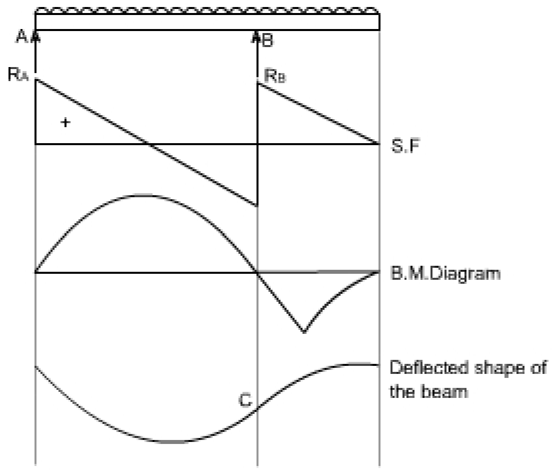
In the problem given below, the intensity of loading varies from q_1 kN/m at one end to the q_2 kN/m at the other end.This problem can be treated by considering a U.d.i of intensity q_1 kN/m over the entire span and a uniformly varying load of 0 to (q_2- q_1)kN/m over the entire span and then super impose teh two loadings.



Point of Contraflexure:

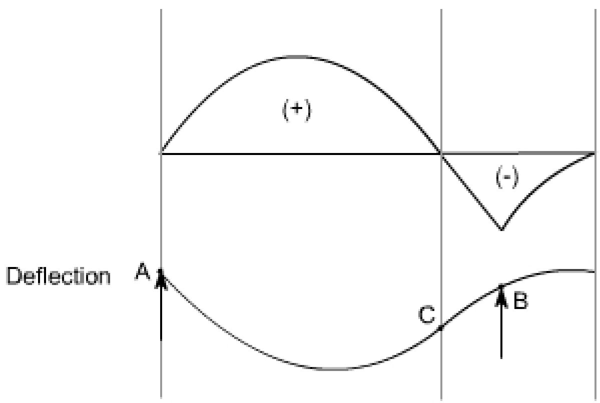


Consider the loaded beam a shown below along with the shear force and Bending moment diagrams for It may be observed that this case, the bending moment diagram is completely positive so that the curvature of the beam varies along its length, but it is always concave upwards or sagging.However if we consider a again a loaded beam as shown below along with the S.F and B.M diagrams, then



It may be noticed that for the beam loaded as in this case,

The bending moment diagram is partly positive and partly negative.If we plot the deflected shape of the beam just below the bending moment



This diagram shows that L.H.S of the beam ‘sags' while the R.H.S of the beam ‘hogs'

The point C on the beam where the curvature changes from sagging to hogging is a point of contraflexure.

OR

It corresponds to a point where the bending moment changes the sign, hence in order to find the point of contraflexures obviously the B.M would change its sign when it cuts the X-axis therefore to get the points of contraflexure equate the bending moment equation equal to zero.The fibre stress is zero at such sections

Note: there can be more than one point of contraflexure.

UNIT -3

SIMPLE BENDING THEORY OR THEORY OF FLEXURE FOR

INITIALLY STRAIGHT BEAMS

(The normal stress due to bending are called flexure stresses)

Preamble:

When a beam having an arbitrary cross section is subjected to a transverse loads the beam will bend. In addition to bending the other effects such as twisting and buckling may occur, and to investigate a problem that includes all the combined effects of bending, twisting and buckling could become a complicated one. Thus we are interested to investigate the bending effects alone, in order to do so, we have to put certain constraints on the geometry of the beam and the manner of loading.

Assumptions:

The constraints put on the geometry would form the **assumptions:**

1. Beam is initially **straight,** and has a **constant cross-section.**

2. Beam is made of **homogeneous material** and the beam has a **longitudinal plane of symmetry.**

3. Resultant of the applied loads lies in the plane of symmetry.

4. The geometry of the overall member is such that bending not buckling is the primary cause of failure.

5. Elastic limit is nowhere exceeded and **‘E'** is same in tension and compression.

6. Plane cross - sections remains plane before and after bending.

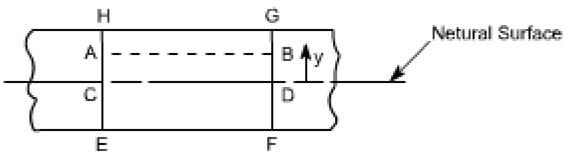


Fig 1(a)

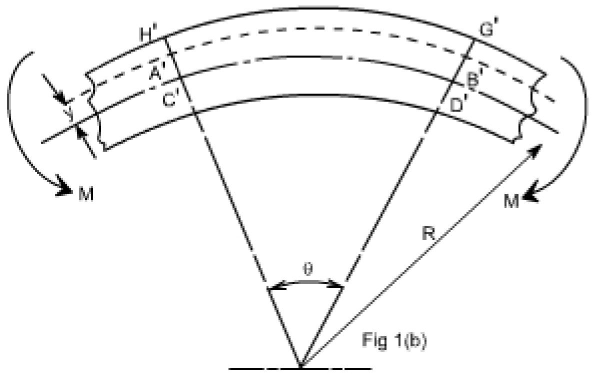


Fig 1(b)

Let us consider a beam initially unstressed as shown in fig 1(a). Now the beam is subjected to a constant bending moment (i.e. ‘Zero Shearing Force') along its length as would be obtained by applying equal couples at each end. The beam will bend to the radius R as shown in Fig 1(b)

As a result of this bending, the top fibers of the beam will be subjected to tension and the bottom to compression it is reasonable to suppose, therefore, **that some where between the two there are points at which the stress is zero. The locus of all such points is known as neutral axis** . The radius of curvature R is then measured to this axis. For symmetrical sections the N. A. is the axis of symmetry but what ever the section N. A. will always pass through the centre of the area or centroid.

The above restrictions have been taken so as to eliminate the possibility of 'twisting' of the beam.

Concept of pure bending:

Loading restrictions:

As we are aware of the fact internal reactions developed on any cross-section of a beam may consists of a resultant normal force, a resultant shear force and a resultant couple. In order to ensure that the bending effects alone are investigated, we shall put a constraint on the loading such that the resultant normal and the resultant shear forces are zero on any cross-section perpendicular to the longitudinal axis of the member,

That means F = 0

since $\frac{dM}{dX} = F = 0$ or M = constant.

Thus, the zero shear force means that the bending moment is constant or the bending is same at every cross-section of the beam. Such a situation may be visualized or envisaged when the beam or some portion of the beam, as been loaded only by pure couples at its ends. It must be recalled that the couples are assumed to be loaded in the plane of symmetry.

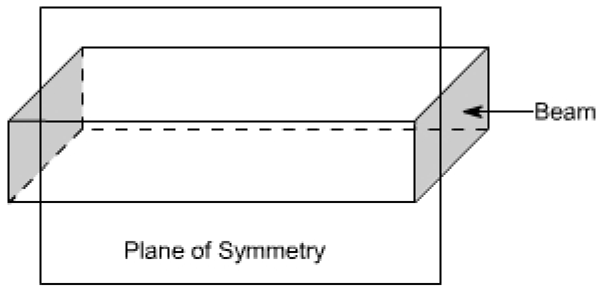


Fig (1)

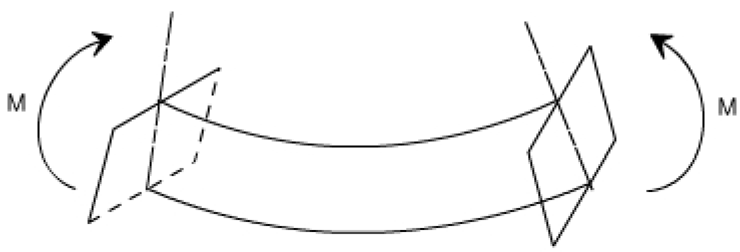
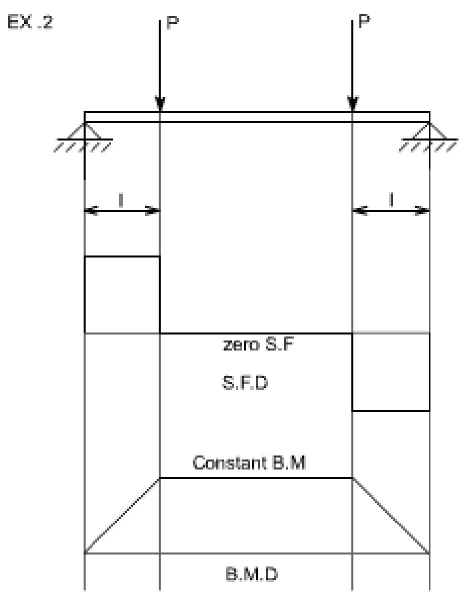
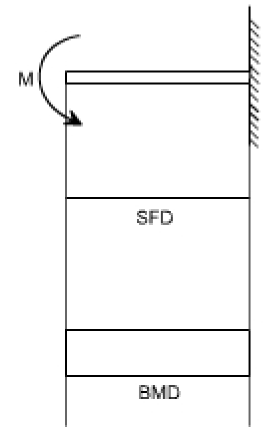


Fig (2)

When a member is loaded in such a fashion it is said to be in **pure bending.** The examples of pure bending have been indicated in EX 1and EX 2 as shown below :



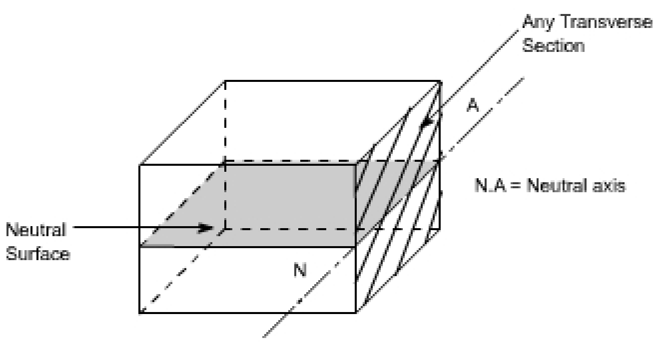
EX. 1



When a beam is subjected to pure bending are loaded by the couples at the ends, certain cross-section gets deformed and we shall have to make out the conclusion that,

1. Plane sections originally perpendicular to longitudinal axis of the beam remain plane and perpendicular to the longitudinal axis even after bending , i.e. the cross-section A'E', B'F' (refer Fig 1(a)) do not get warped or curved.

2. In the deformed section, the planes of this cross-section have a common intersection i.e. any time originally parallel to the longitudinal axis of the beam becomes an arc of circle.



We know that when a beam is under bending the fibres at the top will be lengthened while at the bottom will be shortened provided the bending moment M acts at the ends. In between these there are some fibres which remain unchanged in length that is they are not strained, that is they do not carry any stress. The plane containing such fibres is called neutral surface.

The line of intersection between the neutral surface and the transverse exploratory section is called the neutral axisNeutral axis **(N A)** .

Bending Stresses in Beams or Derivation of Elastic Flexural formula :

In order to compute the value of bending stresses developed in a loaded beam, let us consider the two cross-sections of a beam **HE** and **GF** , originally parallel as shown in fig 1(a).when the beam is to bend it is assumed that these sections remain parallel i.e. **H'E'** and **G'F'** , the final position of the sections, are still straight lines, they then subtend some angle q.

Consider now fiber AB in the material, at adistance y from the N.A, when the beam bends this will stretch to A'B'

Therefore,

$$\text{strain in fibre AB} = \frac{\text{change in length}}{\text{orginal length}}$$

$$= \frac{A'B' - AB}{AB} \qquad \text{But } AB = CD \text{ and } CD = C'D'$$

refer to fig1(a) and fig1(b)

$$\therefore \text{strain} = \frac{A'B' - C'D'}{C'D'}$$

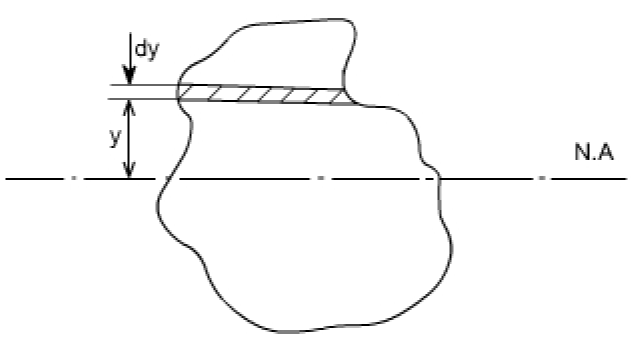
Since CD and C'D' are on the neutral axis and it is assumed that the Stress on the neutral axis zero. Therefore, there won't be any strain on the neutral axis

$$= \frac{(R+y)\theta - R\theta}{R\theta} = \frac{R\theta + y\theta - R\theta}{R\theta} = \frac{y}{R}$$

$$\text{However } \frac{\text{stress}}{\text{strain}} = E \qquad \text{where } E = \text{Young's Modulus of elasticity}$$

Therefore, equating the two strains as obtained from the two relations i.e,

$$\frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \qquad \text{.............(1)}$$



Consider any arbitrary a cross-section of beam, as shown above now the strain on a fibre at a distance 'y' from the N.A, is given by the expression

$$\sigma = \frac{E}{R} y$$

if the shaded strip is of area'dA'
then the force on the strip is

$$F = \sigma\ \delta A = \frac{E}{R} y\ \delta A$$

Moment about the neutral axis would be = $F.y = \frac{E}{R}\ y^2 \delta A$

The toatl moment for the whole
cross-section is therefore equal to

$$M = \sum \frac{E}{R} y^2\ \delta A = \frac{E}{R} \sum y^2 \delta A$$

Now the term $\sum y^2 \delta A$ is the property of the material and is called as a second moment of area of the cross-section and is denoted by a symbol I.

Therefore

$$M = \frac{E}{R} I \qquad ....(2)$$

combining equation 1 and 2 we get

$$\boxed{\frac{\sigma}{y} = \frac{M}{T} = \frac{E}{R}}$$

This equation is known as the Bending Theory Equation. The above proof has involved the assumption of pure bending without any shear force being present. Therefore this termed as the pure bending equation. This equation gives distribution of stresses which are normal to cross-section i.e. in x-direction.

Section Modulus:

From simple bending theory equation, the maximum stress obtained in any cross-section is given as

$$\sigma_{max}{}^{m} = \frac{M}{T} y_{max}{}^{m}$$

For any given allowable stress the maximum moment which can be accepted by a particular shape of cross-section is therefore

$$M = \frac{I}{y_{max}{}^{m}} \sigma_{max}{}^{m}$$

For ready comparison of the strength of various beam cross-section this relationship is some times written in the form

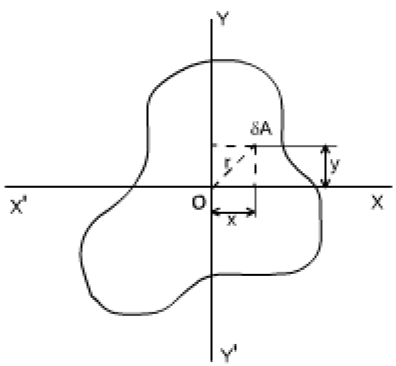
$$M = Z\,\sigma_{max^m} \text{ where } Z = \frac{I}{y_{max^m}}$$ **Is termed as section modulus**

The higher value of Z for a particular cross-section, the higher the bending moment which it can withstand for a given maximum stress.

Theorems to determine second moment of area: There are two theorems which are helpful to determine the value of second moment of area, which is required to be used while solving the simple bending theory equation.

Second Moment of Area :

Taking an analogy from the mass moment of inertia, the second moment of area is defined as the summation of areas times the distance squared from a fixed axis. (This property arised while we were driving bending theory equation). This is also known as the moment of inertia. An alternative name given to this is second moment of area, because the first moment being the sum of areas times their distance from a given axis and the second moment being the square of the distance or $\int y^2\, dA$.



Consider any cross-section having small element of area d A then by the definition

I_x(Mass Moment of Inertia about x-axis) = $\int y^2\, dA$ and I_y(Mass Moment of Inertia about y-axis) = $\int x^2\, dA$

Now the moment of inertia about an axis through 'O' and perpendicular to the plane of figure is called the polar moment of inertia. (The polar moment of inertia is also the area moment of inertia).

i.e,

J = polar moment of inertia

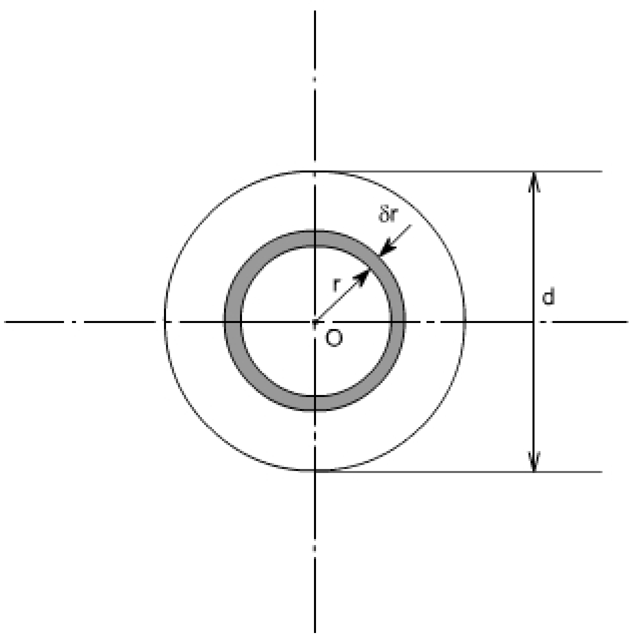
$$= \int r^2 \, dA$$
$$= \int (x^2 + y^2) \, dA$$
$$= \int x^2 \, dA + \int y^2 \, dA$$
$$= I_X + I_Y$$
$$\text{or } J = I_X + I_Y \qquad \text{..........} (1)$$

The relation (1) is known as the **perpendicular axis theorem** and may be stated as follows:

The sum of the Moment of Inertia about any two axes in the plane is equal to the moment of inertia about an axis perpendicular to the plane, the three axes being concurrent, i.e, the three axes exist together.

CIRCULAR SECTION :

For a circular x-section, the polar moment of inertia may be computed in the following manner



Consider any circular strip of thickness dr located at a radius 'r'.

Than the area of the circular strip would be dA = 2pr. dr

$$J = \int r^2 dA$$

Taking the limits of intergration from 0 to d/2

$$J = \int_0^{\frac{d}{2}} r^2 2\pi r \delta r$$

$$= 2\pi \int_0^{\frac{d}{2}} r^3 \delta r$$

$$J = 2\pi \left[\frac{r^4}{4} \right]_0^{\frac{d}{2}} = \frac{\pi d^4}{32}$$

however, by perpendicular axis theorem

$$J = I_x + I_y$$

But for the circular cross-section, the I_x and I_y are both equal being moment of inertia about a diameter

$$I_{dia} = \frac{1}{2} J$$

$$I_{dia} = \frac{\pi d^4}{64}$$

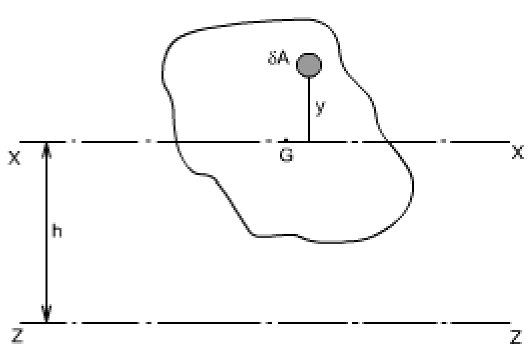
for a hollow circular section of diameter D and d, the values of J and I are defined as

$$J = \frac{\pi (D^4 - d^4)}{32}$$

Thus $$I = \frac{\pi (D^4 - d^4)}{64}$$

Parallel Axis Theorem:

The moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the centroid plus the area times the square of the distance between the axes.



If ‘ZZ' is any axis in the plane of cross-section and ‘XX' is a parallel axis through the centroid G, of the cross-section, then

$$I_z = \int (y + h)^2 \, dA \text{ by definition (moment of inertia about an axis ZZ)}$$

$$= \int (\; + 2yh + h^2) \, dA$$

$$= \int y^2 dA + h^2 \int dA + 2h \int y dA$$

$$\text{Since } \int y dA = 0$$

$$= \int y^2 dA + h^2 \int dA$$

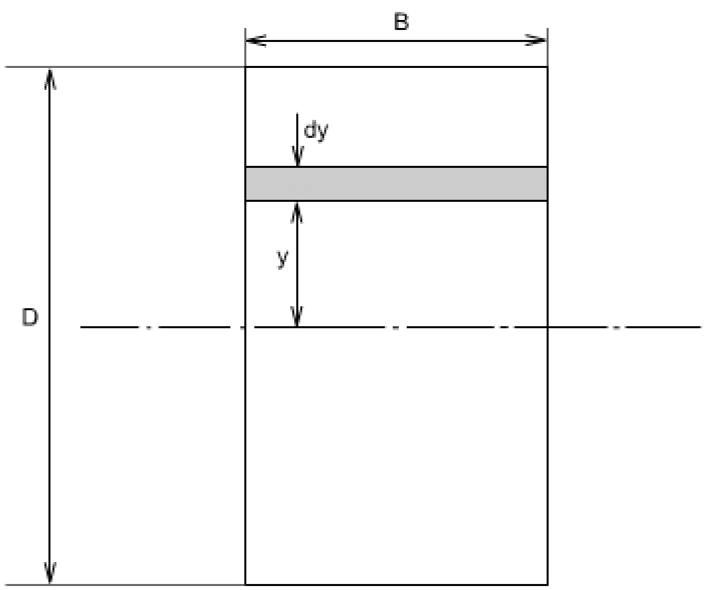
$$= \int y^2 dA + h^2 A$$

$$I_z = I_x + Ah^2 \qquad I_x = I_G \text{ (since cross-section axes also pass through G)}$$

$$\text{Where } A = \text{Total area of the section}$$

Rectangular Section:

For a rectangular x-section of the beam, the second moment of area may be computed as below :



Consider the rectangular beam cross-section as shown above and an element of area **dA** , thickness **dy** , breadth **B** located at a distance **y** from the neutral axis, which by symmetry passes through the centre of section. The second moment of area **I** as defined earlier would be

$$I_{N.A} = \int y^2 dA$$

Thus, for the rectangular section the second moment of area about the neutral axis i.e., an axis through the centre is given by

$$I_{N.A} = \int_{\frac{-D}{2}}^{\frac{D}{2}} y^2 \,(B\,dy)$$

$$= B \int_{\frac{-D}{2}}^{\frac{D}{2}} y^2\,dy$$

$$= B\left[\frac{y^3}{3}\right]_{\frac{D}{2}}^{\frac{-D}{2}}$$

$$= \frac{B}{3}\left[\frac{D^3}{8} - \left(\frac{-D^3}{8}\right)\right]$$

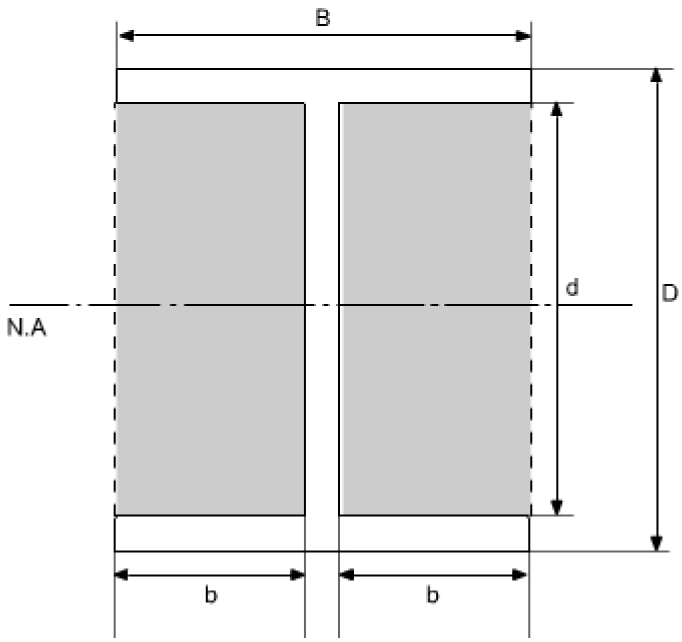
$$= \frac{B}{3}\left[\frac{D^3}{8} + \frac{D^3}{8}\right]$$

$$I_{N.A} = \frac{BD^3}{12}$$

Similarly, the second moment of area of the rectangular section about an axis through the lower edge of the section would be found using the same procedure but with integral limits of **0** to **D** .

Therefore $$I = B\left[\frac{y^3}{3}\right]_0^D = \frac{BD^3}{3}$$

These standards formulas prove very convenient in the determination of I_{NA} for build up sections which can be conveniently divided into rectangles. For instance if we just want to find out the Moment of Inertia of an I - section, then we can use the above relation.



$$I_{N.A} = I_{\text{of dotted rectangle}} - I_{\text{of shaded portion}}$$

$$\therefore\ I_{N.A} = \frac{BD^3}{12} - 2\left(\frac{bd^3}{12}\right)$$

$$I_{N.A} = \frac{BD^3}{12} - \frac{bd^3}{6}$$

Use of Flexure Formula:

Illustrative Problems:

An I - section girder, 200mm wide by 300 mm depth flange and web of thickness is 20 mm is used as simply supported beam for a span of 7 m. The girder carries a distributed load of 5 KN /m and a concentrated load of 20 KN at mid-span.

Determine the

(i). The second moment of area of the cross-section of the girder

(ii). The maximum stress set up.

Solution:

The second moment of area of the cross-section can be determined as follows :

For sections with symmetry about the neutral axis, use can be made of standard I value for a rectangle about an axis through centroid i.e. (bd 3)/12. The section can thus be divided into convenient rectangles for each of which the neutral axis passes through the centroid. Example in the case enclosing the girder by a rectangle

$$I_{girder} = I_{rectangle} - I_{shaded\ portion}$$

$$= \left[\frac{200 \times 300^3}{12}\right] 10^{-12} - 2\left[\frac{90 \times 260^3}{12}\right] 10^{-12}$$

$$= (4.5 - 2.64)10^{-4}$$

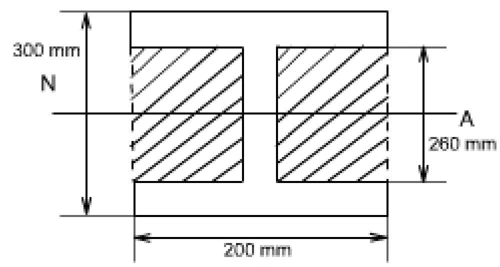
$$= 1.86 \times 10^{-4}\ m^4$$

The maximum stress may be found from the simple bending theory by equation

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

i.e.

$$\sigma_{max^m} = \frac{M_{max^m}}{I}\, y_{max^m}$$



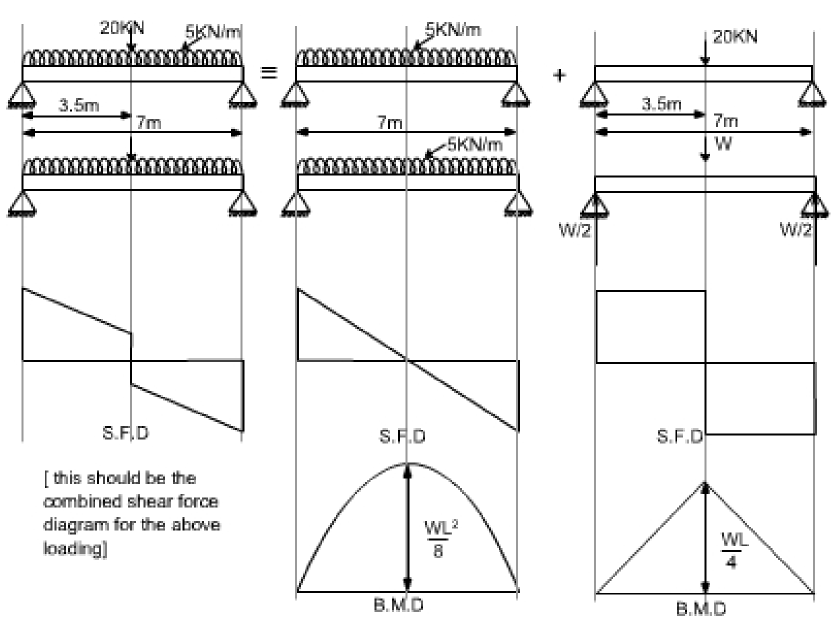
Computation of Bending Moment:

In this case the loading of the beam is of two types

(a) Uniformly distributed load

(b) Concentrated Load

In order to obtain the maximum bending moment the technique will be to consider each loading on the beam separately and get the bending moment due to it as if no other forces acting on the structure and then superimpose the two results.

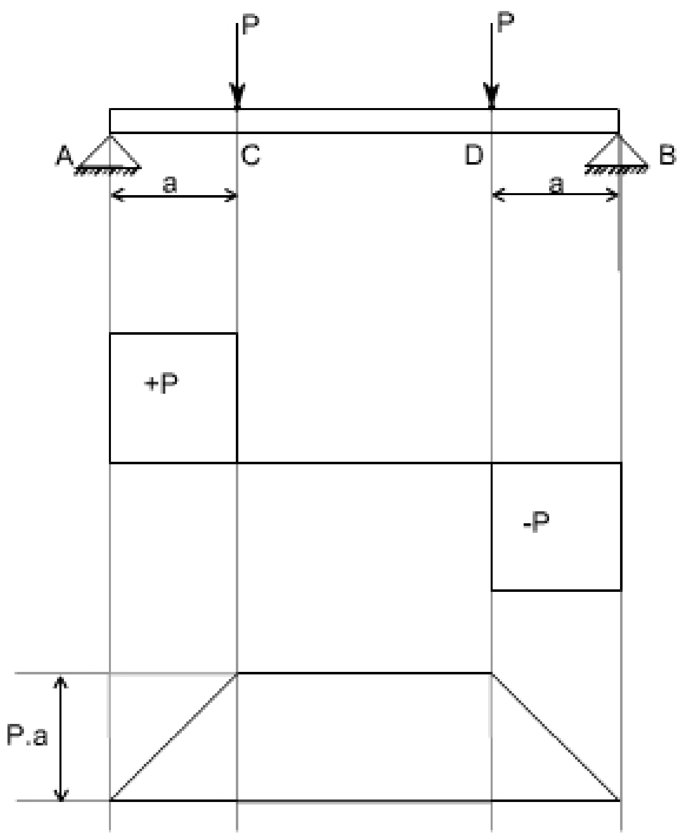


Hence

$$M_{max^m} = \frac{wL}{4} + \frac{wL^2}{8}$$
$$= \frac{20 \times 10^3 \times 7}{4} + \frac{5 \times 10^3 \times 7^2}{8}$$
$$= (35.0 + 30.63)10^3$$
$$= 65.63 \text{ k Nm}$$
$$\sigma_{max^m} = \frac{M_{max^m}}{I} y_{max^m}$$
$$= \frac{65.63 \times 10^3 \times 150 \times 10^3}{1.06 \times 10^{-4}}$$
$$\sigma_{max^m} = 51.8 \text{ MN/m}^2$$

Shearing Stresses in Beams

All the theory which has been discussed earlier, while we discussed the bending stresses in beams was for the case of pure bending i.e. constant bending moment acts along the entire length of the beam.



Let us consider the beam AB transversely loaded as shown in the figure above. Together with shear force and bending moment diagrams we note that the middle potion CD of the beam is free from shear force and that its bending moment. M = P.a is uniform between the portion C and D. This condition is called the pure bending condition.

Since shear force and bending moment are related to each other F= dM/dX (eq) therefore if the shear force changes than there will be a change in the bending moment also, and then this won't be the pure bending.

Conclusions:

Hence one can conclude from the pure bending theory was that the shear force at each X-section is zero and the normal stresses due to bending are the only ones produced.

In the case of non-uniform bending of a beam where the bending moment varies from one X-section to another, there is a shearing force on each X-section and shearing stresses are also induced in the material. The deformation associated with those shearing stresses causes " warping " of the x-section so that the assumption which we assumed while deriving the relation $\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$ that the plane cross-section after bending remains plane is violated. Now due to warping the plane cross=section before bending do not remain plane after bending. This complicates the problem but more elaborate analysis Shows that the normal stresses due to bending, as calculated from the equation $\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$

The above equation gives the distribution of stresses which are normal to the cross-section that is in x-direction or along the span of the beam are not greatly altered by the presence of these shearing stresses. Thus, it is justifiable to use the theory of pure bending in the case of non uniform bending and it is accepted practice to do so.

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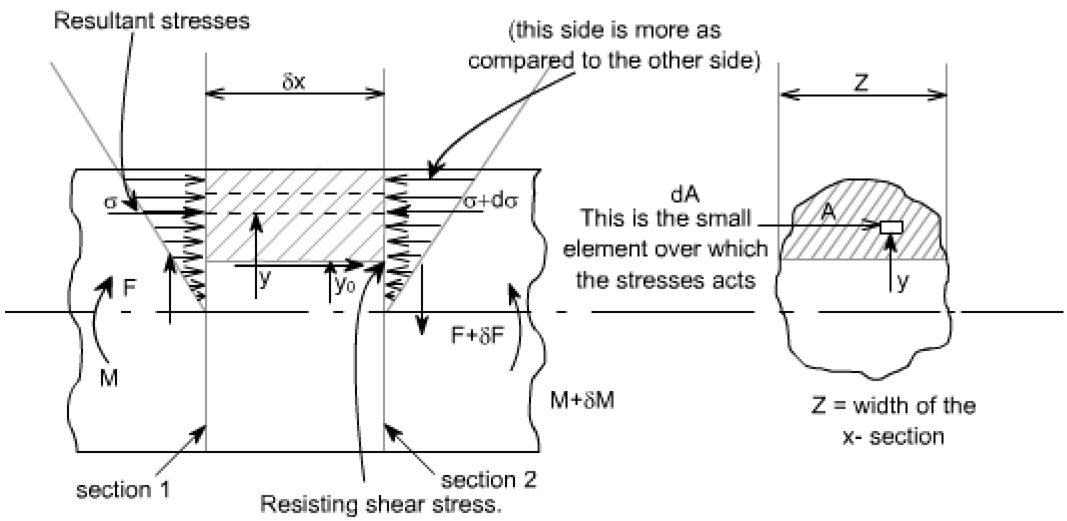
SHEAR STRESSES

Concept of Shear Stresses in Beams:

By the earlier discussion we have seen that the bending moment represents the resultant of certain linear distribution of normal stresses s_x over the cross-section. Similarly, the shear force F_x over any cross-section must be the resultant of a certain distribution of shear stresses.

Derivation of equation for shearing stress:



Assumptions:

1. Stress is uniform across the width (i.e. parallel to the neutral axis)

2. The presence of the shear stress does not affect the distribution of normal bending stresses.

It may be noted that the assumption no.2 cannot be rigidly true as the existence of shear stress will cause a distortion of transverse planes, which will no longer remain plane.

In the above figure let us consider the two transverse sections which are at a distance **'dx'** apart. The shearing forces and bending moments being **F, F + dF** and **M, M + dM** respectively. Now due to the shear stress on transverse planes there will be a complementary shear stress on longitudinal planes parallel to the neutral axis.

Let t be the value of the complementary shear stress (and hence the transverse shear stress) at a distance '**Y'0** from the neutral axis. **Z** is the width of the x-section at this position

A is area of cross-section cut-off by a line parallel to the neutral axis.

$\bar{y}$ = distance of the centroid of Area from the neutral axis.

Let **s , s + ds** are the normal stresses on an element of area **dA** at the two transverse sections, then there is a difference of longitudinal forces equal to (**ds . dA)** , and this quantity summed over the area A is in equilibrium with the transverse shear stress **t** on the longitudinal plane of area z **dx** .

i.e $\tau . z \delta x = \int d\sigma . dA$

from the bending theory equation

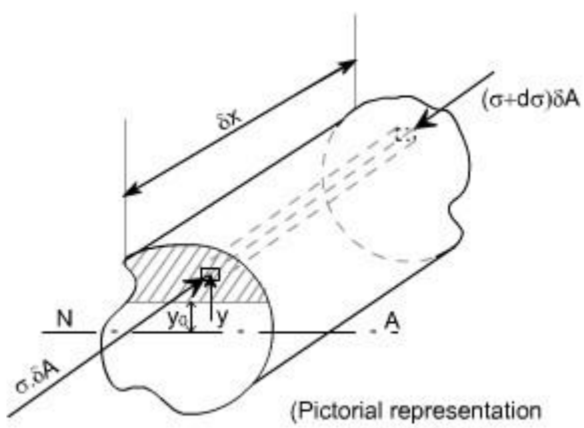
$$\frac{\sigma}{y} = \frac{M}{I}$$

$$\sigma = \frac{M \cdot y}{I}$$

$$\sigma + d\sigma = \frac{(M + \delta M) \cdot y}{I}$$

Thus $$d\sigma = \frac{\delta M . y}{I}$$

The figure shown below indicates the pictorial representation of the part.



(Pictorial representation of entire part)

$$d\sigma = \frac{\delta M . y}{I}$$

$$\tau . z\delta x = \int d\sigma . dA$$

$$= \int \frac{\delta M . y . \delta A}{I}$$

$$\tau . z\delta x = \frac{\delta M}{I} \int y . \delta A$$

But $F = \frac{\delta M}{\delta x}$

i.e. $\tau = \frac{F}{I . z} \int y . \delta A$

But from definition, $\int y . dA = A\bar{y}$

$\int y . dA$ is the first moment of area of the shaded portion

and $\bar{y}$ = centroid of the area 'A'

Hence

So substituting $\tau = \frac{F . A . \bar{y}}{I . z}$

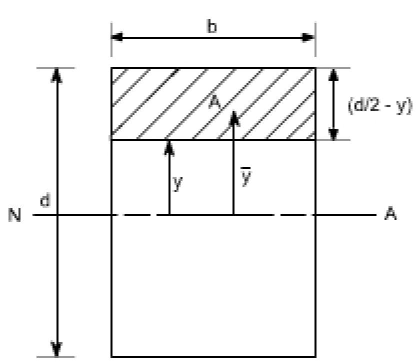
Where 'z' is the actual width of the section at the position where ' t ' is being calculated and I is the total moment of inertia about the neutral axis.

Shearing stress distribution in typical cross-sections:

Let us consider few examples to determine the shear stress distribution in a given X-sections

Rectangular x-section:

Consider a rectangular x-section of dimension b and d



A is the area of the x-section cut off by a line parallel to the neutral $\bar{y}$ is the distance of the centroid of A from the neutral axis

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

for this case, $A = b\left(\frac{d}{2} - y\right)$

While $\bar{y} = \left[\frac{1}{2}\left(\frac{d}{2} - y\right) + y\right]$

i.e $\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$ and $z = b\,;\, I = \frac{b.d^3}{12}$

substituting all these values, in the formula

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

$$= \frac{F.b.\left(\frac{d}{2} - y\right).\frac{1}{2}.\left(\frac{d}{2} + y\right)}{b.\frac{b.d^3}{12}}$$

$$= \frac{\frac{F}{2}.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{\frac{b.d^3}{12}}$$

$$= \frac{6.F.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{b.d^3}$$

This shows that there is a parabolic distribution of shear stress with y.

The maximum value of shear stress would obviously beat the location y = 0.

Such that $\tau_{max} = \frac{6.F}{b.d^3} \cdot \frac{d^2}{4}$

$$= \frac{3.F}{2.b.d}$$

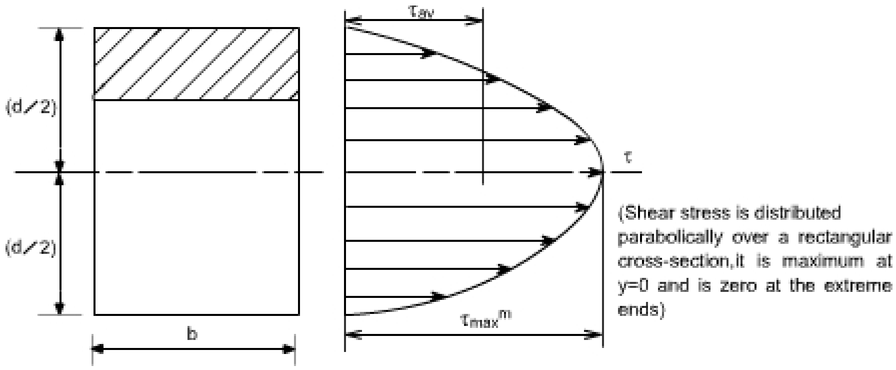
So $\boxed{\tau_{max} = \frac{3.F}{2.b.d}}$ The value of τ_{max} occurs at the neutral axis

The mean shear stress in the beam is defined as

$$\tau_{mean} \text{ or } \tau_{avg} = F/A = F/b.d$$

So $\tau_{max} = 1.5\, \tau_{mean} = 1.5\, \tau_{avg}$

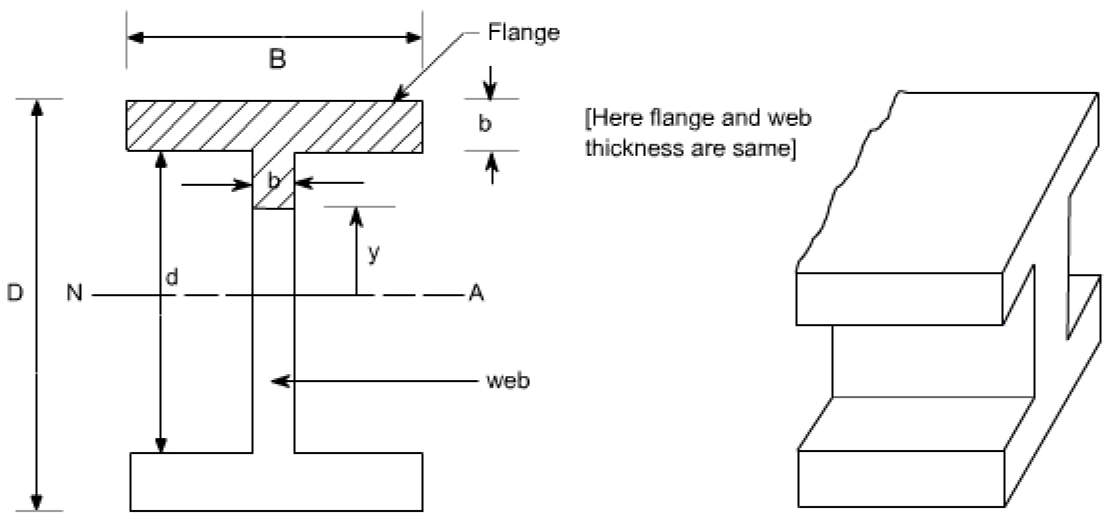
Therefore the shear stress distribution is shown as below.



It may be noted that the shear stress is distributed parabolically over a rectangular cross-section, it is maximum at y = 0 and is zero at the extreme ends.

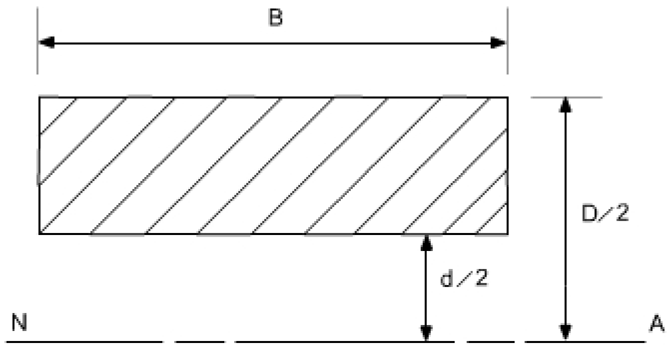
<u>I - section :</u>

Consider an I - section of the dimension shown below.



The shear stress distribution for any arbitrary shape is given as $\tau = \frac{F A \bar{y}}{Z I}$

Let us evaluate the quantity $A\bar{y}$, the $A\bar{y}$ quantity for this case comprise the contribution due to flange area and web area



Flange area

$$\text{Area of the flange} = B\left(\frac{D-d}{2}\right)$$

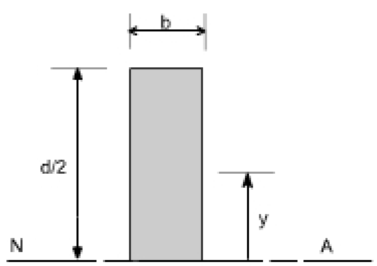
Distance of the centroid of the flange from the N.A

$$\bar{y} = \frac{1}{2}\left(\frac{D-d}{2}\right) + \frac{d}{2}$$

$$\bar{y} = \left(\frac{D+d}{4}\right)$$

Hence,

$$A\bar{y}\Big|_{Flange} = B\left(\frac{D-d}{2}\right)\left(\frac{D-d}{4}\right)$$



Web Area

Area of the web

$$A = b\left(\frac{d}{2} - y\right)$$

Distance of the centroid from N.A

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} - y\right) + y$$

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

Therefore,

$$A\bar{y}\big|_{web} = b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right)$$

Hence,

$$A\bar{y}\big|_{Total} = B\left(\frac{D - d}{2}\right)\left(\frac{D + d}{4}\right) + b\left(\frac{d}{2} - y\right)\left(\frac{d}{2} + y\right)\frac{1}{2}$$

Thus,

$$A\bar{y}\big|_{Total} = B\left(\frac{D^2 - d^2}{8}\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

Therefore shear stress,

$$\tau = \frac{F}{bI}\left[\frac{B\left(D^2 - d^2\right)}{8} + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)\right]$$

To get the maximum and minimum values of t substitute in the above relation.

y = 0 at N. A. And y = d/2 at the tip.

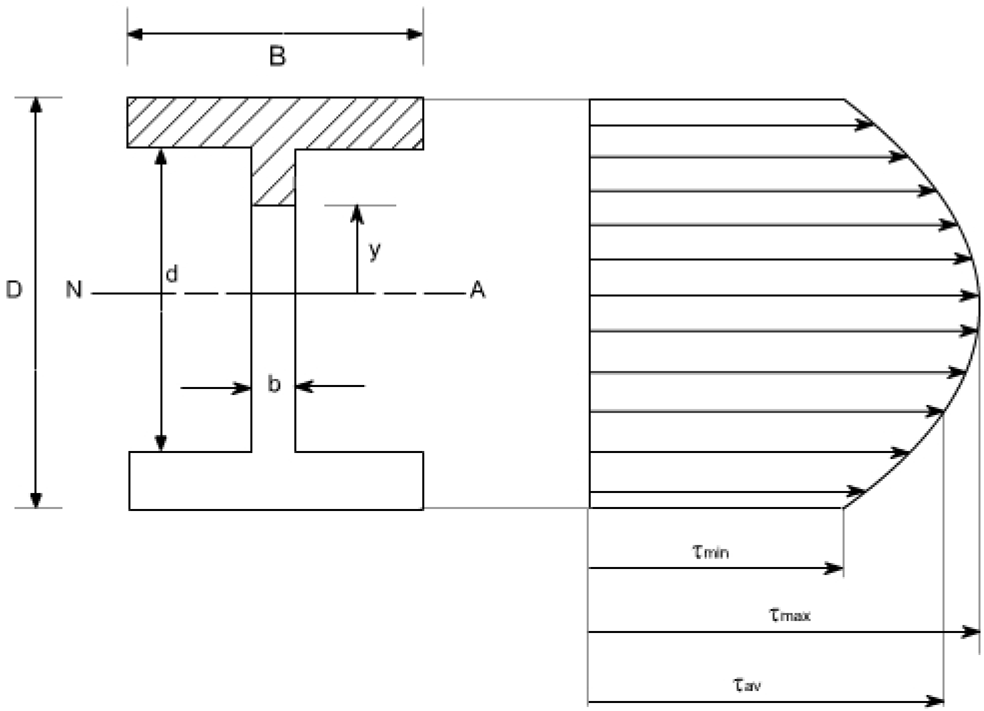
The maximum shear stress is at the neutral axis. i.e. for the condition y = 0 at N. A.

Hence, τ_{max} at $y = 0 = \dfrac{F}{8\,b\,I}\left[B\left(D^2 - d^2\right) + bd^2\right]$ ....(2)

The minimum stress occur at the top of the web, the term bd 2 goes off and shear stress is given by the following expression

τ_{min} at $y = d/2 = \dfrac{F}{8\,b\,I}\left[B\left(D^2 - d^2\right)\right]$ ....(3)

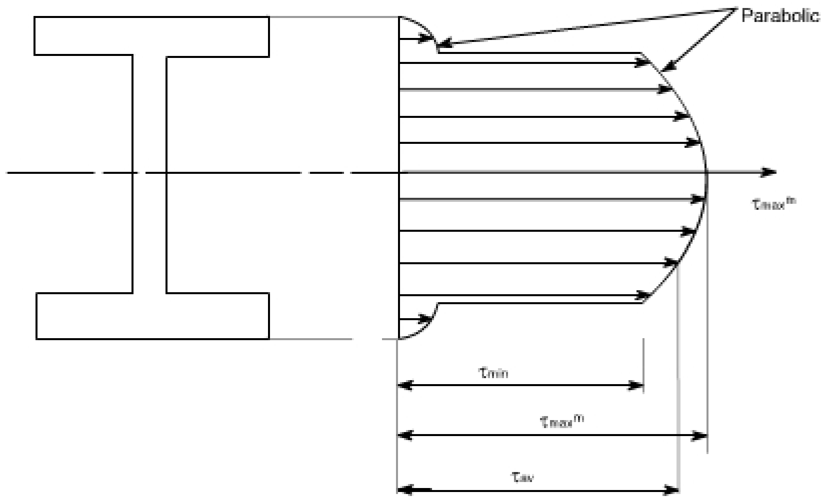
The distribution of shear stress may be drawn as below, which clearly indicates a parabolic distribution



$$\tau_{max^m} = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$$

Note: from the above distribution we can see that the shear stress at the flanges is not zero, but it has some value, this can be analyzed from equation (1). At the flange tip or flange or web interface y = d/2.Obviously than this will have some constant value and than onwards this will have parabolic distribution.

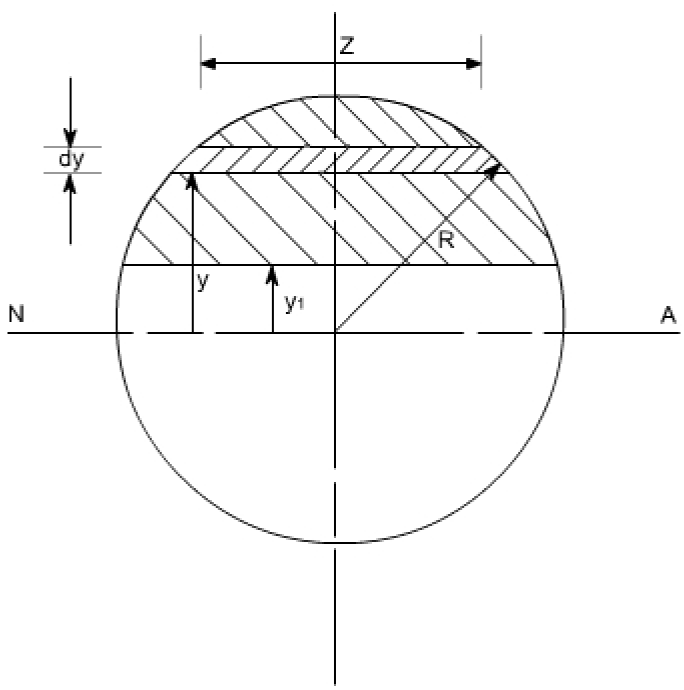
In practice it is usually found that most of shearing stress usually about 95% is carried by the web, and hence the shear stress in the flange is neglible however if we have the concrete analysis i.e. if we analyze the shearing stress in the flange i.e. writing down the expression for shear stress for flange and web separately, we will have this type of variation.



This distribution is known as the “top – hat” distribution. Clearly the web bears the most of the shear stress and bending theory we can say that the flange will bear most of the bending stress.

Shear stress distribution in beams of circular cross-section:

Let us find the shear stress distribution in beams of circular cross-section. In a beam of circular cross-section, the value of Z width depends on y.



Using the expression for the determination of shear stresses for any arbitrary shape or a arbitrary section.

$$\tau = \frac{FA\bar{y}}{ZI} = \frac{FA\int y\,dA}{ZI}$$

Where òy dA is the area moment of the shaded portion or the first moment of area.

Here in this case 'dA' is to be found out using the Pythagoras theorem

$$\left(\frac{Z}{2}\right)^2 + y^2 = R^2$$

$$\left(\frac{Z}{2}\right)^2 = R^2 - y^2 \text{ or } \frac{Z}{2} = \sqrt{R^2 - y^2}$$

$$Z = 2\sqrt{R^2 - y^2}$$

$$dA = Z\,dy = 2.\sqrt{R^2 - y^2}\,.dy$$

$I_{N.A}$ for a circular cross-section $= \frac{\pi R^4}{4}$

Hence,

$$\tau = \frac{FA\bar{y}}{ZI} = \frac{F}{\frac{\pi R^4}{4}2\sqrt{R^2 - y^2}}\int_{y_1}^{R} 2y\sqrt{R^2 - y^2}\,dy$$

Where R = radius of the circle.

[The limits have been taken from y_1 to R because we have to find moment of area the shaded portion]

$$= \frac{4F}{\pi R^4\sqrt{R^2 - y^2}}\int_{y_1}^{R} y\sqrt{R^2 - y^2}\,dy$$

The integration yields the final result to be

$$\tau = \frac{4F\left(R^2 - y_1^2\right)}{3\pi R^4}$$

Again this is a parabolic distribution of shear stress, having a maximum value when $y_1 = 0$

$$\tau_{maxm}|_{y_1 = 0} = \frac{4F}{3\pi R^2}$$

Obviously at the ends of the diameter the value of $y_1 = \pm R$ thus $\tau = 0$

so this again a parabolic distribution; maximum at the neutral axis

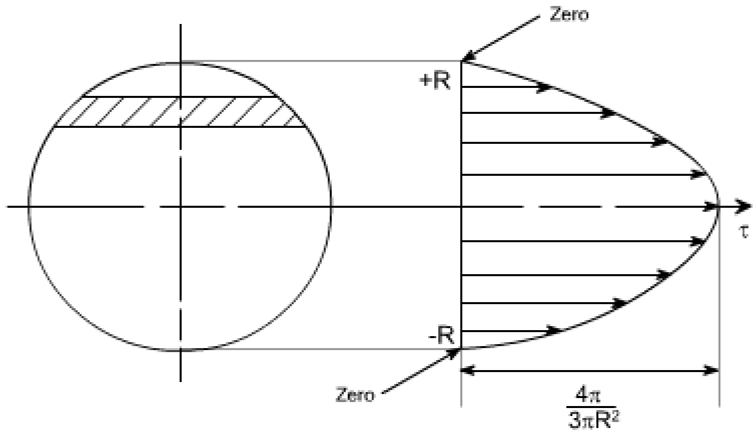
Also

$$\tau_{avg} \text{ or } \tau_{mean} = \frac{F}{A} = \frac{F}{\pi R^2}$$

Hence,

$$\boxed{\tau_{maxm} = \frac{4}{3}\tau_{avg}}$$

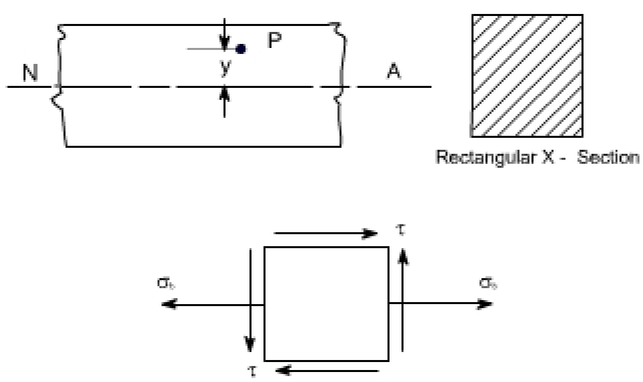
The distribution of shear stresses is shown below, which indicates a parabolic distribution



Principal Stresses in Beams

It becomes clear that the bending stress in beam s_x is not a principal stress, since at any distance y from the neutral axis; there is a shear stress t (or t_{xy} we are assuming a plane stress situation)

In general the state of stress at a distance y from the neutral axis will be as follows.



At some point 'P' in the beam, the value of bending stresses is given as

$\sigma_b = \frac{My}{I}$ for a beam of rectangular cross - section of dimensions b and d; $I = \frac{bd^3}{12}$

$\sigma_b = \frac{12\,My}{bd^3}$

whereas the value shear stress in the rectangular cross - section is given as

$$\tau = \frac{6F}{bd^3}\left[\frac{d^2}{4} - y^2\right]$$

Hence the values of principle stress can be determined from the relations,

$$\sigma_1, \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau_{xy}^2}$$

Letting $\sigma_y = 0$; $\sigma_x = \sigma_b$,the values of σ_1 and σ_2 can be computed as

$$\text{Hence } \sigma_1 / \sigma_2 = \frac{1}{2}\left(\frac{12My}{bd^3}\right) \pm \frac{1}{2}\sqrt{\left(\frac{12\,My}{bd^3}\right)^2 + 4\left(\frac{6F}{bd^3}\left(\frac{d^2}{4} - y^2\right)\right)^2}$$

$$\sigma_1, \sigma_2 = \frac{6}{bd^3}\left[My \pm \sqrt{\left\{M^2 y^2 + F^2\left(\frac{d^2}{4} - y^2\right)^2\right\}}\right]$$

Also,

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \text{putting } \sigma_y = 0$$

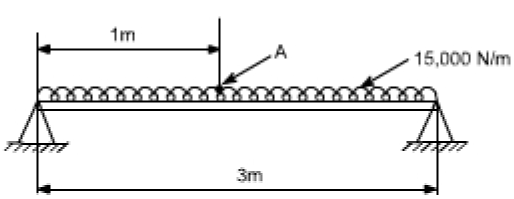
we get,

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x}$$

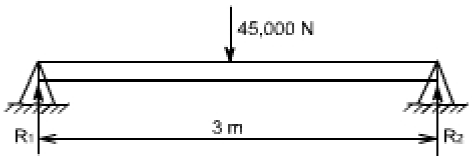
After substituting the appropriate values in the above expression we may get the inclination of the principal planes.

Illustrative examples: Let us study some illustrative examples, pertaining to determination of principal stresses in a beam

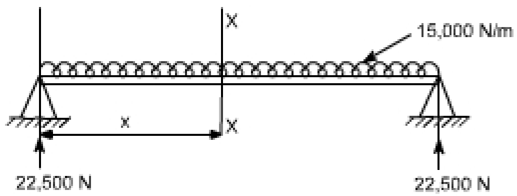
1. Find the principal stress at a point A in a uniform rectangular beam 200 mm deep and 100 mm wide, simply supported at each end over a span of 3 m and carrying a uniformly distributed load of 15,000 N/m.



Solution: The reaction can be determined by symmetry



$R_1 = R_2 = 22{,}500$ N



consider any cross-section X-X located at a distance x from the left end.

Hence,

S. F $_{at\ XX}$ =22,500 – 15,000 x

B.M $_{at\ XX}$ = 22,500 x – 15,000 x (x/2) = 22,500 x – 15,000 . x^2 / 2

Therefore,

S. F $_{at\ X}$ = 1 m = 7,500 N

B. M $_{at\ X}$ = 1 m = 15,000 N

$$S.F|_{x=1m} = 7{,}500\text{ N}$$

$$B.M|_{x=1m} = 15{,}000\text{ N.m}$$

$$\sigma_x = \frac{My}{I}$$

$$= \frac{15{,}000 \times 5 \times 10^{-2} \times 12}{10 \times 10^{-12} \times \left(20 \times 10^{-2}\right)^3}$$

$$\sigma_x = 11.25\text{ MN/m}^2$$

For the compution of shear stresses

$$\tau = \frac{6F}{bd^3}\left[\frac{d^2}{4} - y^2\right] \qquad \text{putting } y = 50\text{ mm}, d = 200\text{ mm}$$

$$F = 7500\text{ N}$$

$$\tau = 0.422\text{ MN/m}^2$$

Now substituting these values in the principal stress equation,

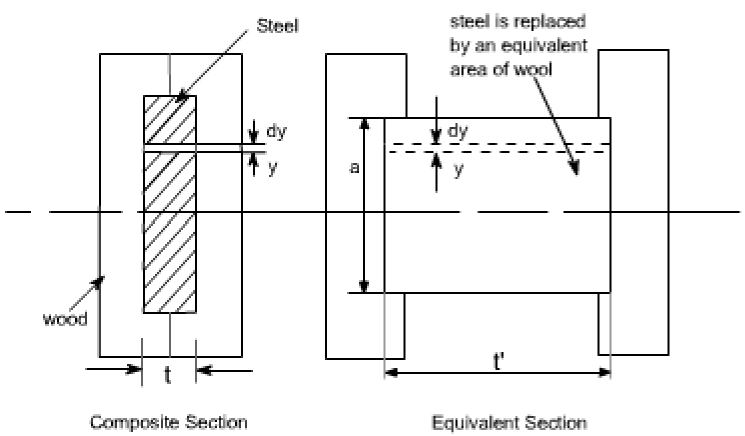
We get $s_1 = 11.27$ MN/m^2

s$_2$ = - 0.025 MN/m^2

Bending Of Composite or Flitched Beams

A composite beam is defined as the one which is constructed from a combination of materials. If such a beam is formed by rigidly bolting together two timber joists and a reinforcing steel plate, then it is termed as a flitched beam.

The bending theory is valid when a constant value of Young's modulus applies across a section it cannot be used directly to solve the composite-beam problems where two different materials, and therefore different values of E, exists. The method of solution in such a case is to replace one of the materials by an equivalent section of the other.



Consider, a beam as shown in figure in which a steel plate is held centrally in an appropriate recess/pocket between two blocks of wood .Here it is convenient to replace the steel by an equivalent area of wood, retaining the same bending strength. i.e. the moment at any section must be the same in the equivalent section as in the original section so that the force at any given dy in the equivalent beam must be equal to that at the strip it replaces.

$$\sigma.t = \sigma'.t' \text{ or } \boxed{\frac{\sigma}{\sigma'} = \frac{t'}{t}}$$

recalling $\sigma = E.\varepsilon$

Thus

$$\varepsilon E t = \varepsilon' E' t'$$

Again, for true similarity the strains must be equal,

$$\varepsilon = \varepsilon' \text{ or } E t = E' t' \text{ or } \boxed{\frac{E}{E'} = \frac{t'}{t}}$$

Thus, $\boxed{t' = \frac{E}{E'}.t}$

Hence to replace a steel strip by an equivalent wooden strip the thickness must be multiplied by the modular ratio E/E'.

The equivalent section is then one of the same materials throughout and the simple bending theory applies. The stress in the wooden part of the original beam is found directly and that in the steel found from the value at the same point in the equivalent material as follows by utilizing the given relations.

$$\frac{\sigma'}{\sigma} = \frac{t'}{t}$$

$$\frac{\sigma'}{\sigma} = \frac{E'}{E}$$

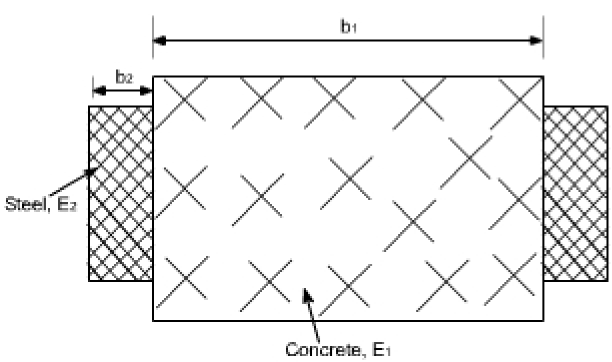
Stress in steel = modular ratio x stress in equivalent wood

The above procedure of course is not limited to the two materials treated above but applies well for any material combination. The wood and steel flitched beam was nearly chosen as a just for the sake of convenience.

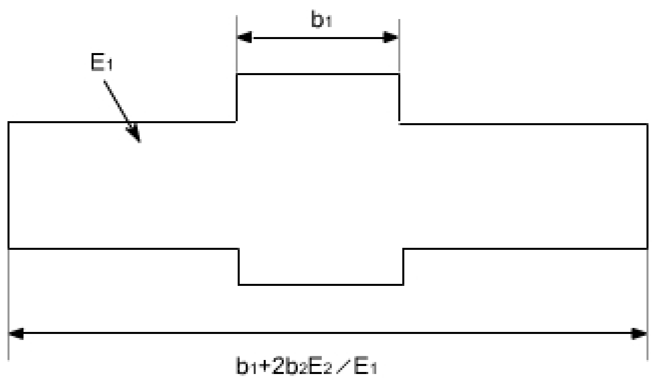
Assumption

In order to analyze the behavior of composite beams, we first make the assumption that the materials are bonded rigidly together so that there can be no relative axial movement between them. This means that all the assumptions, which were valid for homogenous beams are valid except the one assumption that is no longer valid is that the Young's Modulus is the same throughout the beam.

The composite beams need not be made up of horizontal layers of materials as in the earlier example. For instance, a beam might have stiffening plates as shown in the figure below.



Again, the equivalent beam of the main beam material can be formed by scaling the breadth of the plate material in proportion to modular ratio. Bearing in mind that the strain at any level is same in both materials, the bending stresses in them are in proportion to the Young's modulus.



Members Subjected to Combined Loads

Combined Bending & Twisting : In some applications the shaft are simultaneously subjected to bending moment M and Torque T.The Bending moment comes on the shaft due to gravity or Inertia loads. So the stresses are set up due to bending moment and Torque.

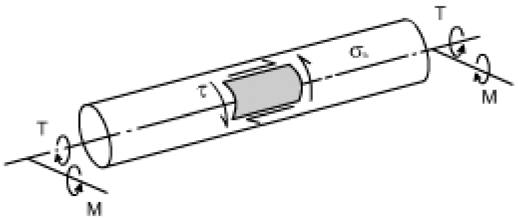
For design purposes it is necessary to find the principal stresses, maximum shear stress, which ever is used as a criterion of failure.

From the simple bending theory equation $\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$

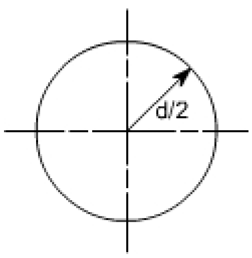
If s_b is the maximum bending stresses due to bending.

$$\sigma_b = \frac{M.y}{I}$$

$$\sigma_b \Big|_{max^m} = \frac{M}{I}.y\,max^m$$



For the case of circular shafts $y_{max}{}^{m}$ – equal to d/2 since y is the distance from the neutral axis.



I is the moment of inertia for circular shafts

I = pd^4 /64

Hence then, the maximum bending stresses developed due to the application of bending moment M is

$$\sigma_b\Big|_{max^m} = \frac{M}{\pi d^4/64}\cdot\frac{d}{2}$$

$$\sigma_b\Big|_{max^m} = \frac{32M}{\pi d^3} \qquad (1)$$

From the torsion theory, the maximum shear stress on the surface of the shaft is given by the torsion equation

$$\frac{T}{J} = \frac{\tau'}{r} = \frac{G.\theta}{L}$$

$$\Rightarrow \frac{\tau'}{r} = \frac{T}{J}$$

Where t' is the shear stress at any radius r but when the maximum value is desired the value of r should be maximum and the value of r is maximum at r = d/2

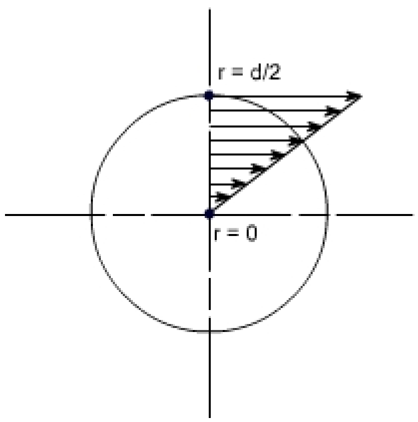
Thus $\tau_{max^m} = \frac{T}{J}.\frac{d}{2}$

$$J = \frac{\pi d^4}{32}$$

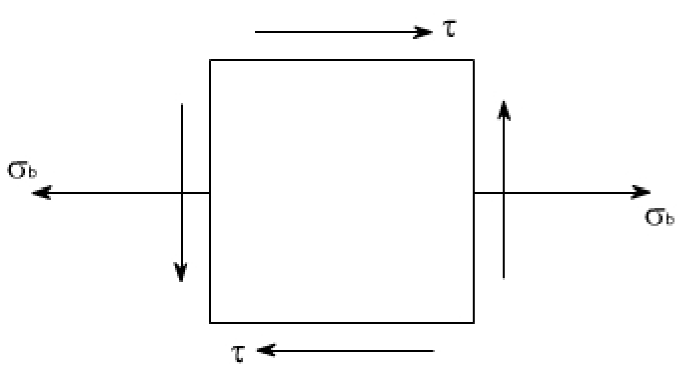
substituting the value of J, we get

$$\tau_{max^m} = \frac{16T}{\pi d^3} \qquad (2)$$

The nature of the shear stress distribution is shown below :



This can now be treated as the two – dimensional stress system in which the loading in a vertical plane in zero i.e. $s_y = 0$ and $s_x = s_b$ and is shown below :



Thus, the principle stresses may be obtained as

$$\sigma_1, \sigma_2 = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

or

$$\sigma_1 = \frac{\sigma_b}{2} + \frac{1}{2}\sqrt{\sigma_b^2 + 4\tau_{max}^2}$$

$$= \frac{32M}{\pi d^3 .2} + \frac{1}{2}\sqrt{\left(\frac{32M}{\pi d^3}\right)^2 + 4\left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \frac{16M}{\pi d^3 2} + \frac{1}{2}\sqrt{\left(\frac{32M}{\pi d^3}\right)^2 + \left(\frac{2.16T}{\pi d^3}\right)^2}$$

$$= \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right]$$

Equivalent Bending Moment :

Now let us define the term the equivalent bending moment which acting alone, will produce the same maximum principal stress or bending stress.Let M_e be the equivalent bending moment, then due to bending

$$\sigma_1 = \frac{32M_e}{\pi d^3}$$

Futher

$$\sigma_1 = \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right]$$

Thus, equating the two we get

$$\boxed{M_e = \frac{1}{2}\left[M + \sqrt{M^2 + T^2}\right]}$$

Equivalent Torque :

At we here already proved that s $_1$ and s $_2$ for the combined bending and twisting case are expressed by the relations:

$$\sigma_1, \sigma_2 = \frac{16}{\pi d^3}\left\{M \pm \sqrt{M^2 + T^2}\right\}$$

$$\text{or } \sigma_1 = \frac{16}{\pi d^3}\left[\left\{M + \sqrt{M^2 + T^2}\right\}\right]$$

$$\sigma_2 = \frac{16}{\pi d^3}\left[\left\{M - \sqrt{M^2 + T^2}\right\}\right]$$

$$\text{As } \tau_{max^m} = \frac{\sigma_1 - \sigma_2}{2}$$

$$\text{so } \tau_{max^m} = \frac{16}{\pi d^3}\left[\left\{M + \sqrt{M^2 + T^2}\right\}\right] - \frac{16}{\pi d^3}\left[\left\{M - \sqrt{M^2 + T^2}\right\}\right] \Big/ 2$$

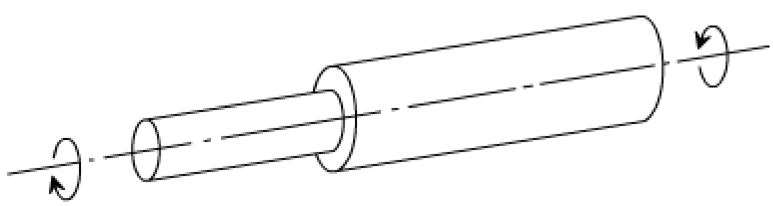
$$\tau_{max^m} = \frac{16}{\pi d^3}\sqrt{M^2 + T^2} = \frac{16}{\pi d^3}.T_e$$

where $\sqrt{M^2 + T^2}$ is defined as the equivalent torque, which acting alone would produce the same maximum shear stress as produced by the pure torsion

Thus, $\boxed{T_e = \sqrt{M^2 + T^2}}$

Composite shafts: (in series)

If two or more shaft of different material, diameter or basic forms are connected together in such a way that each carries the same torque, then the shafts are said to be connected in series & the composite shaft so produced is therefore termed as series – connected.



Here in this case the equilibrium of the shaft requires that the torque 'T' be the same through out both the parts.

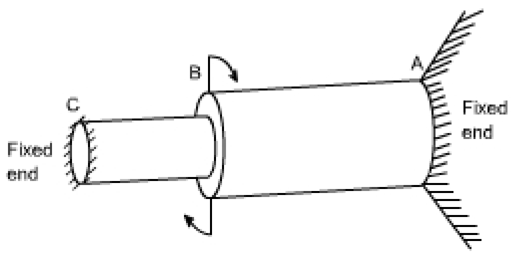
In such cases the composite shaft strength is treated by considering each component shaft separately, applying the torsion – theory to each in turn. The composite shaft will therefore be as weak as its weakest component. If relative dimensions of the various parts are required then a solution is usually effected by equating the torque in each shaft e.g. for two shafts in series

$$T = \frac{G_1 J_1 \theta_1}{L_1} = \frac{G_2 J_2 \theta_2}{L_2}$$

In some applications it is convenient to ensure that the angle of twist in each shaft are equal i.e. $q_1 = q_2$, so that for similar materials in each shaft $\frac{J_1}{L_1} = \frac{J_2}{L_2}$ or $\frac{L_1}{L_2} = \frac{J_1}{J_2}$

The total angle of twist at the free end must be the sum of angles $q_1 = q_2$ over each x - section

Composite shaft parallel connection: If two or more shafts are rigidly fixed together such that the applied torque is shared between them then the composite shaft so formed is said to be connected in parallel.



For parallel connection.

Total Torque $T = T_1 + T_2$

In this case the angle of twist for each portion are equal and $\frac{T_1L_1}{G_1J_1} = \frac{T_2L_2}{G_2J_2}$

for equal lengths(as is normaly the case for parallel shafts) $\frac{T_1}{T_2} = \frac{G_1J_1}{G_2J_2}$

This type of configuration is statically indeterminate, because we do not know how the applied torque is apportioned to each segment, To deal such type of problem the procedure is exactly the same as we have discussed earlier,

Thus two equations are obtained in terms of the torques in each part of the composite shaft and the maximun shear stress in each part can then be found from the relations.

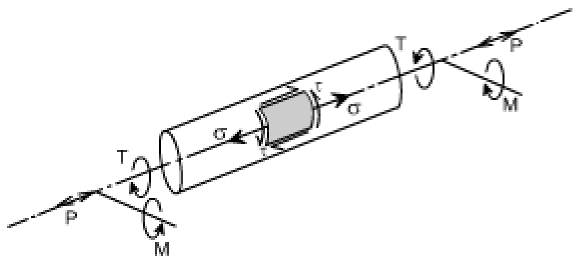
$$\tau_1 = \frac{T_1R_1}{J_1}$$

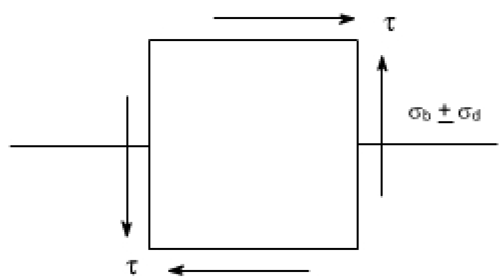
$$\tau_2 = \frac{T_2R_2}{J_2}$$

Combined bending, Torsion and Axial thrust:

Sometimes, a shaft may be subjected to a combined bending, torsion and axial thrust. This type of situation arises in turbine propeller shaft

If P = Thrust load





Then s $_d$ = P / A (stress due to thrust)

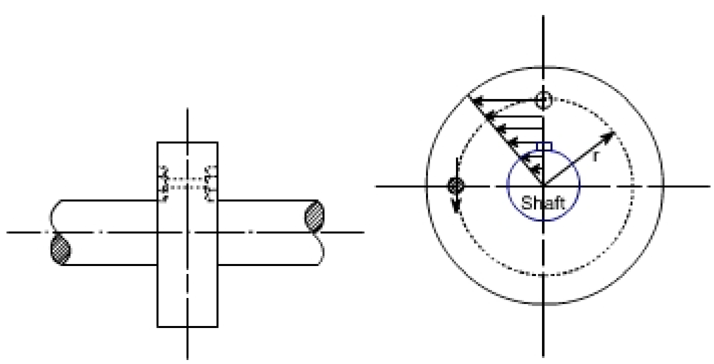
where s_d is the direct stress depending on the whether the steam is tensile on the whether the stress is tensile or compressive

This type of problem may be analyzed as discussed in earlier case.

Shaft couplings: In shaft couplings, the bolts fail in shear. In this case the torque capacity of the coupling may be determined in the following manner

<u>Assumptions:</u>

The shearing stress in any bolt is assumed to be uniform and is governed by the distance from its center to the centre of coupling.



Thus, the torque capacity of the coupling is given as $T = \left(\frac{\pi}{4} d_b^2\right).\tau_b{}'.r.n$

where

d_b = diameter of bolt

t'_b = maximum shear stress in bolt

n = no. of bolts

r = distance from center of bolt to center of coupling

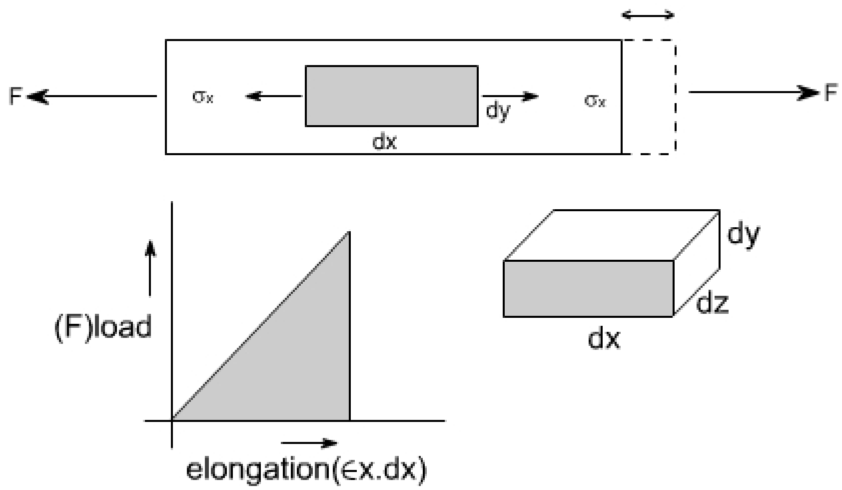
THEORIES OF ELASTIC FAILURE

While dealing with the design of structures or machine elements or any component of a particular machine the physical properties or chief characteristics of the constituent materials are usually found from the results of laboratory experiments in which the components are subject to the simple stress conditions. The most usual test is a simple tensile test in which the value of stress at yield or fracture is easily determined.

However, a machine part is generally subjected simultaneously to several different types of stresses whose actions are combined therefore, it is necessary to have some basis for determining the allowable working stresses so that failure may not occur. Thus, the function of the theories of elastic failure is to predict from the behavior of materials in a simple tensile test when elastic failure will occur under any conditions of applied stress.

A number of theories have been proposed for the brittle and ductile materials.

Strain Energy: The concept of strain energy is of fundamental importance in applied mechanics. The application of the load produces strain in the bar. The effect of these strains is to increase the energy level of the bar itself. Hence a new quantity called strain energy is defined as the energy absorbed by the bar during the loading process. This strain energy is defined as the work done by load provided no energy is added or subtracted in the form of heat. Some times strain energy is referred to as internal work to distinguish it from external work ‘W'. Consider a simple bar which is subjected to tensile force F, having a small element of dimensions dx, dy and dz.



The strain energy U is the area covered under the triangle

$$U = \frac{1}{2}F.\in_x .d_x$$

$$= \frac{1}{2}\sigma_x.dydz.dx\in_x$$

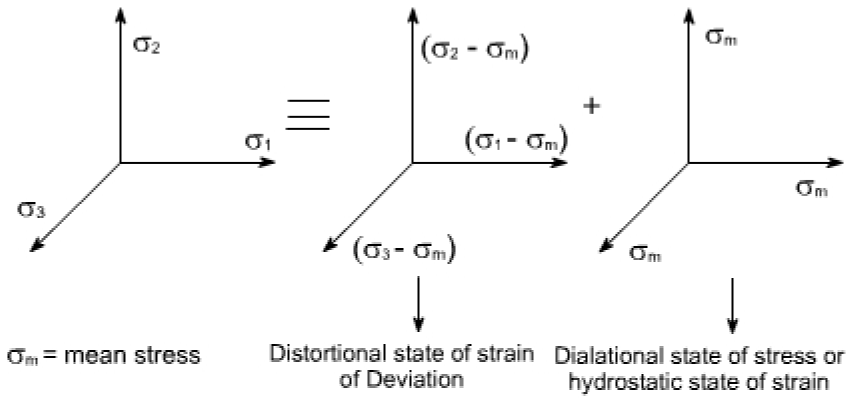
$$= \frac{1}{2}\sigma_x.\in_x .dxdydz$$

$$= \frac{1}{2}\sigma_x\left(\frac{\sigma_x}{E}\right).dxdydz$$

$$\boxed{\frac{U}{volume} = \frac{1}{2}\frac{\sigma_x^2}{E}}$$

A three dimension state of stress respresented by s_1, s_2 and s_3 may be throught of consisting of two distinct state of stresses i.e Distortional state of stress

Hydrostatic state of stresses.



$$\sigma_m = \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{3}$$

Thus, The energy which is stored within a material when the material is deformed is termed as a strain energy. The total strain energy U_r

$U_T = U_d + U_H$

U_d is the strain energy due to the Deviatoric state of stress and U_H is the strain energy due to the Hydrostatic state of stress. Futher, it may be noted that the hydrostatic state of stress results in change of volume whereas the deviatoric state of stress results in change of shape.

THEORIES OF FAILURE

These are five different theories of failures which are generally used

(a) Maximum Principal stress theory (due to Rankine)

(b) Maximum shear stress theory (Guest - Tresca)

(c) Maximum Principal strain (Saint - venant) Theory

(d) Total strain energy per unit volume (Haigh) Theory

(e) Shear strain energy per unit volume Theory (Von – Mises & Hencky)

In all these theories we shall assume.

s_{Yp} = stress at the yield point in the simple tensile test.

s_1, s_2, s_3 - the three principal stresses in the three dimensional complex state of stress systems in order of magnitude.

(A)Maximum Principal stress theory :

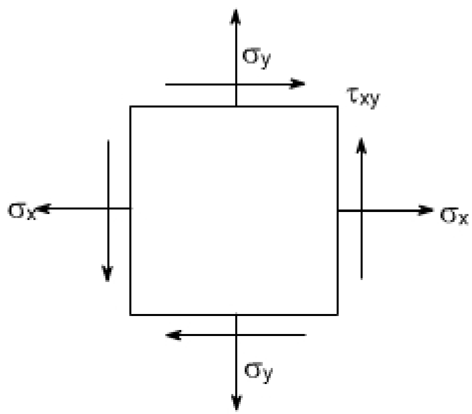
This theory assume that when the maximum principal stress in a complex stress system reaches the elastic limit stress in a simple tension, failure will occur.

Therefore the criterion for failure would be

$s_1 = s_{yp}$

For a two dimensional complex stress system s_1 is expressed as

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4.\tau_{xy}^2}$$
$$= \sigma_{yp}$$

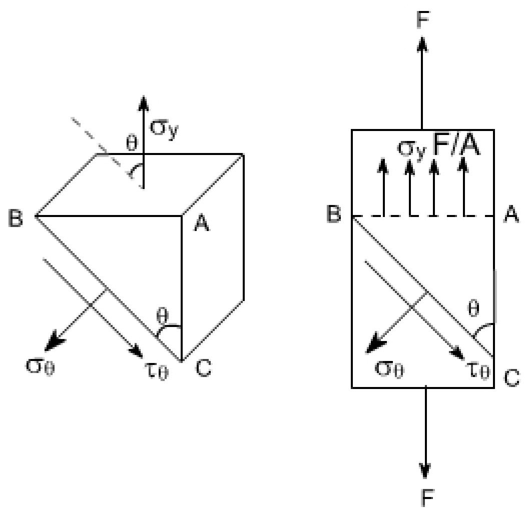


Where s_x, s_y and t_{xy} are the stresses in the any given complex stress system.

b)Maximum shear stress theory:

This theory states that teh failure can be assumed to occur when the maximum shear stress in the complex stress system is equal to the value of maximum shear stress in simple tension.

The criterion for the failure may be established as given below:



For a simple tension case

$$\sigma_\theta = \sigma_y \sin^2\theta$$

$$\tau_\theta = \frac{1}{2}\sigma_y \sin 2\theta$$

$$\tau_\theta\big|_{max^m} = \frac{1}{2}\sigma_y \quad \text{or}$$

$$\tau_{max^m} = \frac{1}{2}\sigma_{yp}$$

whereas for the two dimentional complex stress system

$$\tau_{max^m} = \left(\frac{\sigma_1 - \sigma_2}{2}\right)$$

where σ_1 = maximum principle stress

σ_2 = minimum principal stress

$$\text{so} \quad \frac{\sigma_1 - \sigma_2}{2} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2 xy}$$

$$\frac{\sigma_1 - \sigma_2}{2} = \frac{1}{2}\sigma_{yp} \Rightarrow \sigma_1 - \sigma_2 = \sigma_y$$

$$\Rightarrow \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2 xy} = \sigma_{yp}$$

becomes the criterion for the failure.

c. Maximum Principal strain theory :

This Theory assumes that failure occurs when the maximum strain for a complex state of stress system becomes equals to the strain at yield point in the tensile test for the three dimensional complex state of stress system.

For a 3 - dimensional state of stress system the total strain energy U_t per unit volume in equal to the total work done by the system and given by the equation

$U_t = 1/2\sigma_1 \in_1 + 1/2\sigma_2 \in_2 + 1/2\sigma_3 \in_3$

substituting the values of $\in_1, \in_2$ and $\in_3$

$$\in_1 = \frac{1}{E}\left[\sigma_1 - \gamma(\sigma_2 + \sigma_3)\right]$$

$$\in_2 = \frac{1}{E}\left[\sigma_2 - \gamma(\sigma_1 + \sigma_3)\right]$$

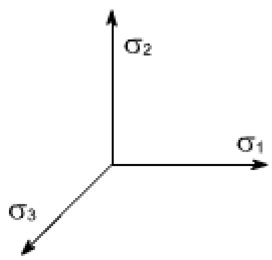
$$\in_3 = \frac{1}{E}\left[\sigma_3 - \gamma(\sigma_1 + \sigma_2)\right]$$

Thus, the failure criterion becomes

$$\left(\frac{\sigma_1}{E} - \gamma\frac{\sigma_2}{E} - \gamma\frac{\sigma_3}{E}\right) = \frac{\sigma_{yp}}{E}$$

or

$$\boxed{\sigma_1 - \gamma\sigma_2 - \gamma\sigma_3 = \sigma_{yp}}$$



d. Total strain energy per unit volume theory :

The theory assumes that the failure occurs when the total strain energy for a complex state of stress system is equal to that at the yield point a tensile test.

$$\frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\gamma(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] = \frac{\sigma_{yp}^2}{2E}$$

Therefore, the failure criterion becomes $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\gamma(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_{yp}^2$

It may be noted that this theory gives fair by good results for ductile materials.

e. Maximum shear strain energy per unit volume theory :

This theory states that the failure occurs when the maximum shear strain energy component for the complex state of stress system is equal to that at the yield point in the tensile test.

$$\frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = \frac{\sigma_{yp}^2}{6G}$$

Where G = shear modulus of regidity

Hence the criterion for the failure becomes $\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = 2\sigma_{yp}^2$

As we know that a general state of stress can be broken into two components i.e,

(i) Hydrostatic state of stress (the strain energy associated with the hydrostatic state of stress is known as the volumetric strain energy)

(ii) Distortional or Deviatoric state of stress (The strain energy due to this is known as the shear strain energy)

As we know that the strain energy due to distortion is given as

$$U_{distortion} = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

This is the distortion strain energy for a complex state of stress, this is to be equaled to the maximum distortion energy in the simple tension test. In order to get we may assume that one of the principal stress say (s_1) reaches the yield point (s_{yp}) of the material. Thus, putting in above equation $s_2 = s_3 = 0$ we get distortion energy for the simple test i.e

$$U_d = \frac{2\sigma_1^2}{12G}$$

Futher $\sigma_1 = \sigma_{yp}$

Thus, $\boxed{U_d = \frac{\sigma_{yp}^2}{6G}}$ for a simple tension test.

Members Subjected to Axisymmetric Loads

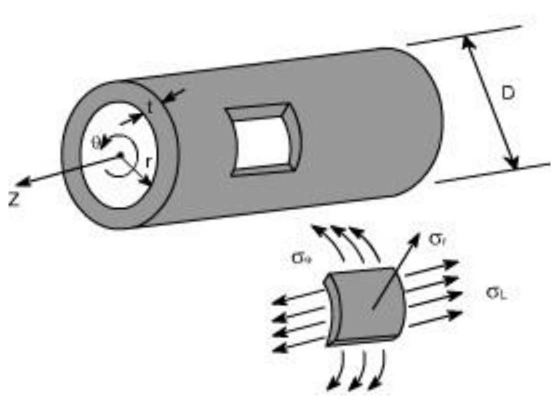
Pressurized thin walled cylinder:

Preamble : Pressure vessels are exceedingly important in industry. Normally two types of pressure vessel are used in common practice such as cylindrical pressure vessel and spherical pressure vessel.

In the analysis of this walled cylinders subjected to internal pressures it is assumed that the radial plans remains radial and the wall thickness dose not change due to internal pressure. Although the internal pressure acting on the wall causes a local compressive stresses (equal to pressure) but its value is neglibly small as compared to other stresses & hence the sate of stress of an element of a thin walled pressure is considered a biaxial one.

Further in the analysis of them walled cylinders, the weight of the fluid is considered neglible.

Let us consider a long cylinder of circular cross - section with an internal radius of R $_2$ and a constant wall thickness't' as showing fig.



This cylinder is subjected to a difference of hydrostatic pressure of ‘p' between its inner and outer surfaces. In many cases, ‘p' between gage pressure within the cylinder, taking outside pressure to be ambient.
By thin walled cylinder we mean that the thickness'' is very much smaller than the radius R_i and we may quantify this by stating than the ratio t / R_i of thickness of radius should be less than 0.1.
An appropriate co-ordinate system to be used to describe such a system is the cylindrical polar one r, q , z shown, where z axis lies along the axis of the cylinder, r is radial to it and q is the angular co-ordinate about the axis.
The small piece of the cylinder wall is shown in isolation, and stresses in respective direction have also been shown.

Type of failure:

Such a component fails in since when subjected to an excessively high internal pressure. While it might fail by bursting along a path following the circumference of the cylinder. Under normal circumstance it fails by circumstances it fails by bursting along a path parallel to the axis. This suggests that the hoop stress is significantly higher than the axial stress.
In order to derive the expressions for various stresses we make following

Applications:

Liquid storage tanks and containers, water pipes, boilers, submarine hulls, and certain air plane components are common examples of thin walled cylinders and spheres, roof domes.

ANALYSIS : In order to analyze the thin walled cylinders, let us make the following assumptions :

- There are no shear stresses acting in the wall.
- The longitudinal and hoop stresses do not vary through the wall.
- Radial stresses s_r which acts normal to the curved plane of the isolated element are neglibly small as compared to other two stresses especially when $\left[\frac{t}{R_i} < \frac{1}{20}\right]$

The state of tress for an element of a thin walled pressure vessel is considered to be biaxial, although the internal pressure acting normal to the wall causes a local compressive stress equal to the internal pressure, Actually a state of tri-axial stress exists on the inside of the vessel. However, for then walled pressure vessel the third stress is much smaller than the other two stresses and for this reason in can be neglected.

Thin Cylinders Subjected to Internal Pressure:

When a thin – walled cylinder is subjected to internal pressure, three mutually

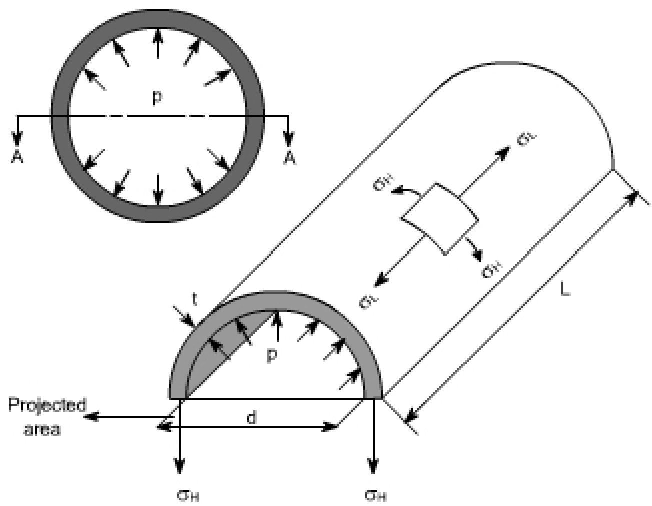
Perpendicular principal stresses will be set up in the cylinder materials, namely

- Circumferential or hoop stress
- The radial stress
- Longitudinal stress

let us define these stresses and determine the expressions for them

Hoop or circumferential stress:

This is the stress which is set up in resisting the bursting effect of the applied pressure and can be most conveniently treated by considering the equilibrium of the cylinder.



In the figure we have shown a one half of the cylinder. This cylinder is subjected to an internal pressure p.

i.e. p = internal pressure

d = inside diameter

L = Length of the cylinder

t = thickness of the wall

Total force on one half of the cylinder owing to the internal pressure 'p'

= p x Projected Area

= p x d x L

= p .d. L ------- (1)

The total resisting force owing to hoop stresses s_H set up in the cylinder walls

= 2 .s_H.L.t ---------(2)

Because s_H.L.t. is the force in the one wall of the half cylinder.

the equations (1) & (2) we get

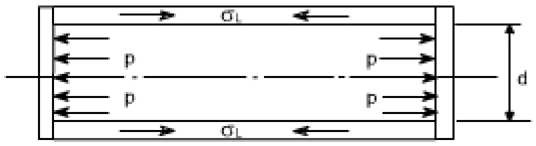
2 . s_H . L . t = p . d . L

s_H = (p . d) / 2t

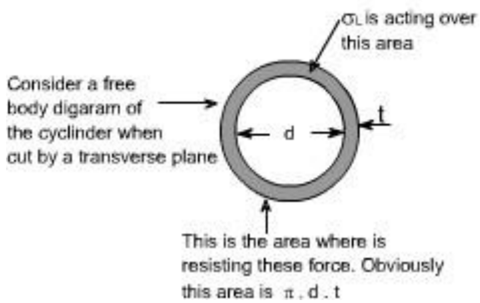
Circumferential or hoop Stress (s_H) = (p .d)/ 2t

Longitudinal Stress:

Consider now again the same figure and the vessel could be considered to have closed ends and contains a fluid under a gage pressure p.Then the walls of the cylinder will have a longitudinal stress as well as a ciccumferential stress.



Total force on the end of the cylinder owing to internal pressure
= pressure x area
= p x p d^2 /4
Area of metal resisting this force = pd.t. (approximately)
because pd is the circumference and this is multiplied by the wall thickness



Hence the longitudnal stresses

$$\text{Set up} = \frac{\text{force}}{\text{area}} = \frac{[p \times \pi d^2/4]}{\pi dt}$$

$$= \frac{pd}{4t} \quad \text{or} \quad \sigma_L = \frac{pd}{4t}$$

or alternatively from equilibrium conditions

$$\sigma_L.(\pi dt) = p.\frac{\pi d^2}{4}$$

$$\text{Thus } \boxed{\sigma_L = \frac{pd}{4t}}$$