

ORDINARY DIFFERENTIAL EQUATIONS AND VECTOR CALCULUS

B.Tech. I Year II Sem.

L T P C
3 1 0 4

Pre-requisites: Mathematical Knowledge at pre-university level

Course Objectives: To learn

1. Methods of solving the differential equations of first order.
2. Methods of solving the differential equations of higher order.
3. Concept, properties of Laplace transforms. Solving ordinary differential equations using Laplace transforms techniques.
4. The physical quantities involved in engineering field related to vector valued functions
5. The basic properties of vector valued functions and their applications to line, surface and volume integrals

Course Outcomes: After learning the contents of this paper the student must be able to

1. Identify whether the given differential equation of first order is exact or not.
2. Solve higher differential equation and apply the concept of differential equation to real world problems.
3. Use the Laplace transforms techniques for solving ODE's.
4. Evaluate the line, surface and volume integrals.
5. Converting line, surface and volume integrals from one to another

Unit: I: First Order ODE

Exact differential equations, Equations reducible to exact differential equations, linear and Bernoulli's equations, Orthogonal Trajectories (only in Cartesian Coordinates). Applications: Newton's law of cooling, Law of natural growth and decay.

Unit: II : Ordinary Differential Equations of Higher Order

Second order linear differential equations with constant coefficients: Non-Homogeneous terms of the type e^{ax} , $\sin$, $\cos ax$, polynomials in x , $e^{ax}V(x)$ and $x V(x)$, method of variation of parameters, Equations reducible to linear ODE with constant coefficients: Legendre's equation, Cauchy-Euler equation. Applications: Electric Circuits

Unit: III: Laplace transforms

Laplace Transforms: Laplace Transform of standard functions, First shifting theorem, Second shifting theorem, Unit step function, Dirac delta function, Laplace transforms of functions when they are multiplied and divided by 't', Laplace transforms of derivatives and integrals of function, Evaluation of integrals by Laplace transforms, Laplace transform of periodic functions, Inverse Laplace transform by different methods, convolution theorem (without proof). Applications: solving Initial value problems by Laplace Transform method.

Unit: IV: Vector Differentiation

Vector point functions and scalar point functions, Gradient, Divergence and Curl, Directional derivatives, Tangent plane and normal line, Vector Identities, Scalar potential functions, Solenoidal and Irrotational vectors.

Unit: V: Vector Integration

Line, Surface and Volume Integrals, Theorems of Green, Gauss and Stokes (without proofs) and their applications.

Text Books:

1. B.S. Grewal, Higher Engineering Mathematics, Khanna Publishers, 36th Edition, 2010
2. R.K. Jain and S.R.K. Iyengar, Advanced Engineering Mathematics, Narosa Publications, 5th Edition, 2016.

Reference Books:

1. Erwin Kreyszig, Advanced Engineering Mathematics, 9th Edition, John Wiley & Sons, 2006.
2. G.B. Thomas and R.L. Finney, Calculus and Analytic geometry, 9th Edition, Pearson, Reprint, 2002.
3. H. K. Dass and Er. Rajnish Verma, Higher Engineering Mathematics, S Chand and Company Limited, New Delhi.
4. N.P. Bali and Manish Goyal, A text book of Engineering Mathematics, Laxmi Publications, Reprint, 2008.

First Order Ordinary differential Equations. 5.5

- Exact differential equation

- Equation reducible to exact
linear & Bernoulli's DE

- Equation reducible to LDE

- Applications:-

1. Newton's law of cooling
2. Law of natural growth and decay
3. Orthogonal trajectories

⇒ Differentiation :- The rate of change of one variable with respect to other variable is called differentiation.

In these case the changing variable is called dependent variable and the other variable is called independent variable

from $\frac{dy}{dx}$ is a differentiation where

y is dependent variable
x is independent variable

⇒ Differential Equation :- An equation which contains differential coefficients or differentials is called as a differential equation

$$\frac{dy}{dx} = \frac{2x}{3y}, \quad \frac{d^2y}{dx^2} + 2y = e^x$$

⇒ Order of a differential equation :-

An order of highest derivative in the differential equation is known as order of a differential equation

(or)

→ A differential equation is said to be of order n if n^{th} derivative is the highest derivative in that equation.

$$\frac{d^2y}{dx^2} + xy = e^x \quad \therefore \text{Order} = 2$$

(In this equation second derivative is the highest derivative)

$$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + xy \quad \therefore \text{Order} = 2$$

(In this second derivative is the highest derivative)

⇒ Degree of a differential equation :-

The highest power of highest order derivative is called that degree of the differential equation (which is free from Rationals & Radicals)

$$\text{Ex: } \left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + xy = 0$$

$\therefore$ Order = 2 In this second
Degree = 3 order is the highest
derivative

$$\left(\frac{d^2y}{dx^2}\right)^{4/3} + \left(\frac{dy}{dx}\right)^6 = 1$$

$$\left(\frac{d^2y}{dx^2}\right)^{4/3} = 1 - \left(\frac{dy}{dx}\right)^6$$

$$\left(\frac{d^2y}{dx^2}\right)^4 = \left[1 - \left(\frac{dy}{dx}\right)^6\right]^3$$

order = 2
degree = 4

Types of differential Equations :-

Differential Equations are separated in two types

1. Ordinary differential Equation (ODE)
2. partial differential Equation (PDE)

ODE :- A differential equation in which the derivatives are with respect to one independent variable is known as Ordinary Differential Equation

Ex: $\frac{dy}{dx} = \frac{2x}{3y}$

$$\frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + 2y = 0$$

PDE :- A differential equation in which the derivatives are with respect to more than one independent variable is known as PDE

Ex: $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$

Formation of a Ordinary differential Equation :-

An Ordinary differential equation can be formed by elimination of an arbitrary constant from the given equation

If the equation contain n arbitrary constants then by differentiating it n times successively and elimination of arbitrary constants from $n+1$ equations (including given equation) we get n^{th} Order (of the given differential equation)

NOTE :- The order of the given differential equation is the no. of arbitrary constants to be eliminated

Ex: Consider $y = A \cos x + B \sin x$ — (1)

where A, B are arbitrary constants

Diff eqⁿ (1) w. r. to x

$$\frac{dy}{dx} = -A (\sin x) + B (\cos x) \text{ — (2)}$$

Diff eqⁿ (2) w. r. to x

$$\frac{d^2y}{dx^2} = -A \cos x + B (-\sin x)$$

$$\frac{d^2y}{dx^2} = -[A \cos x + B \sin x]$$

$$\frac{d^2y}{dx^2} = -y$$

$$\frac{d^2y}{dx^2} + y = 0$$

Is the required differential equation of equation (1).

Solution :-

— Any relation between dependent and Independent variables not containing their derivatives which satisfies the given differential equation is known as Solution

— There are three types of solutions.

1. General solution
2. particular solution
3. Singular solution.

1. General solution :- is a solution of differential equation in which no. of arbitrary constants is equal to order of given differential equation.
2. particular solution :- A solution obtained from the general solution by giving some particular values for arbitrary constants.
3. singular solution :- A solution which is neither general solution nor particular solution.

First order & First degree ODE

- A differential equation which contains first order and first degree derivatives of y and known functions of x and y is known as ordinary differential equation of first order & first degree.

The general form of first order & first degree DE can be written as $\frac{dy}{dx} = f(x, y)$ (or)

$$M(x, y) dx + N(x, y) dy = 0$$

The above differential equation can be solved using the following forms (or) methods.

1. Variable-separable
2. Homogeneous & non-homogeneous
3. Exact DE of equation reducible to exact
4. Linear DE of equation reducible to linear
5. Bernoulli's DE

1. Variable-separable method :-

If the differential equation $\frac{dy}{dx} = f(x, y)$ can be expressed in the form of $-f(x) dx + g(y) dy = 0$ this is known as Variable-separable form.

∴ Its general solution is $\int f(x)dx + \int g(y)dy = c$
(or)

If in an equation it is possible to collect all functions of x and dx on one side and all functions of y and dy on the other side then the variables are said to be separable.

① Solve $\frac{dy}{dx} = e^{x+2y}$

Given that $\frac{dy}{dx} = e^{x+2y}$

$$\frac{dy}{dx} = e^x \cdot e^{2y}$$

$$\frac{dy}{e^{2y}} = e^x dx$$

$$e^{-2y} dy = e^x dx \quad \therefore \int e^{-2y} dy - \int e^x dx = 0$$

on integration

$$\left(\frac{e^{-2y}}{-2} \right) =$$

$$\int e^{-2y} dy - \int e^x dx = c$$

$$\frac{e^{-2y}}{-2} - e^x = c$$

$$e^{-2y} + 2e^x = 2c$$

$$e^{-2y} + 2e^x = k$$

② Solve $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$

Given that

$$\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$$

$$\frac{dy}{dx} = \frac{e^{3x}}{e^{2y}} + x^2 e^{-2y}$$

$$\frac{dy}{dx} = \frac{e^{3x} + x^2}{e^{2y}}$$

$$\frac{dy}{dx} = e^{3x} e^{-2y} + x^2 e^{-2y}$$

$$\frac{dy}{dx} = e^{-2y} [e^{3x} + x^2]$$

$$\frac{dy}{e^{-2y}} = e^{3x} + x^2 dx$$

$$e^{2y} dy = e^{3x} + x^2 dx$$

$$e^{2y} dy - e^{3x} + x^2 dx = 0 \rightarrow \int e^{2y} dy - \int e^{3x} + x^2 dx = C$$

$$\frac{e^{2y}}{2} - \frac{e^{3x}}{3} + \frac{x^3}{3} = C$$

$$\frac{3e^{2y} - 2e^{3x} + 2x^3}{6} = C$$

$$3e^{2y} - 2e^{3x} + 2x^3 = 6C$$

$$3e^{2y} - 2e^{3x} + 2x^3 = 6C$$

⑧

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y}$$

Given that $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y}$

$$\frac{y}{\sqrt{1-y^2}} dy = dx$$

$$\int \frac{1}{\sqrt{t}} dy = \int dt + C$$

put $1-y^2 = t$

$$-2y dy = dt$$

$$y dy = -\frac{dt}{2}$$

$$\int \frac{1}{\sqrt{t}} \left(-\frac{dt}{2}\right) = \int dx + C$$

$$\frac{-1}{2} \int t^{-1/2} dt = x + C$$

$$\frac{-1}{2} \left[\frac{t^{-1/2+1}}{-1/2+1} \right] = x + C$$

$$\frac{-1}{2} \frac{t^{1/2}}{1/2} = x + C$$

$$\Rightarrow -1 t^{1/2} = x + C$$

$$-(1-y^2)^{1/2} = x + C$$

$$x + (1-y^2)^{1/2} = C$$

(4) $\frac{dy}{dx} = \frac{x^2+3}{y^2+1}$

Given that $\frac{dy}{dx} = \frac{x^2+3}{y^2+1}$

$$(y^2+1)dy = (x^2+3)dx$$

$$\int (y^2+1)dy = \int (x^2+3)dx = C$$

$$\frac{y^3}{3} + y - \left[\frac{x^4}{4} + 3x \right] = C$$

$$\frac{y^3 + 3y}{3} - \left[\frac{x^4 + 12x}{4} \right] = C$$

$$\frac{4y^3 + 12y}{12} - \frac{3x^4 + 36x}{12} = C$$

$$4y^3 + 12y - 3x^4 + 36x = 12C$$

Solve $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$

$$e^x \tan y \, dx = -(1 - e^x) \sec^2 y \, dy$$

$$e^x \tan y \, dx = (e^x - 1) \sec^2 y \, dy$$

$$\frac{e^x}{e^x - 1} \, dx = \frac{\sec^2 y}{\tan y} \, dy$$

$$\int \frac{e^x}{e^x - 1} \, dx - \int \frac{\sec^2 y}{\tan y} \, dy = C$$

$$\int \frac{f'(x)}{f(x)} \, dx = \log f(x)$$

Take $e^x - 1 = t$ In this integral of

Take $\tan y = z$ In the RHS.

$$e^x \, dx = dt$$

$$\sec^2 y \, dy = dz$$

$$\therefore \int \frac{dt}{t} =$$

$$\int \frac{dt}{t} - \int \frac{dz}{z} = C$$

$$\log t - \log z = \log C$$

$$\log(e^x - 1) - \log(\tan y) = \log C$$

$$\log \left[\frac{e^x - 1}{-t \operatorname{tany}} \right] = \log c$$

$$\frac{e^x - 1}{-t \operatorname{tany}} = c$$

$$-t \operatorname{tany} = \frac{e^x - 1}{c}$$

$$-t \operatorname{tany} = k(e^x - 1)$$

⑥ Solve $y - x \left(\frac{dy}{dx} \right) = a \left(y^2 + \frac{dy}{dx} \right)$

Given that

$$y - x \frac{dy}{dx} = ay^2 + a \frac{dy}{dx}$$

$$y - ay^2 = a \frac{dy}{dx} + x \frac{dy}{dx}$$

$$y - ay^2 = (a + x) \frac{dy}{dx}$$

$$\frac{y - ay^2}{a + x} = \frac{dy}{dx}$$

$$\frac{dx}{a + x} = \frac{dy}{(y - ay^2)}$$

$$\int \frac{dx}{a + x} = \int \frac{dy}{y(1 - ay)} + c$$

$$\int \frac{1}{a + x} dx = \int \frac{1}{y(1 - ay)} + c$$

$$\log(a + x) = \int \left[\frac{1}{y} - \frac{(a)}{(1 - ay)} \right] dy + c$$

$$\log(a+x) = \log y - \log(1-ay) + \log C$$

$$(\log(a+x) - \log y + \log(1-ay) = \log C$$

$$\log\left[\frac{a+x}{y}\right] + \log(1-ay) = \log C$$

$$\log\left[\frac{a+x}{y}\right] = \log(1-ay)$$

$$\log(a+x) = \log\left[\frac{y}{1-ay}\right] + \log C$$

$$\log(a+x) = \log\left[\frac{cy}{1-ay}\right]$$

$$a+x = \frac{cy}{1-ay}$$

$$(1+x^2) \frac{dy}{dx} = 1+y^2$$

$$\text{Given that } (1+x^2) \frac{dy}{dx} = 1+y^2$$

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

$$\int \frac{1}{1+y^2} dy = \int \frac{1}{1+x^2} dx + C$$

$$-\tan^{-1}(y) = -\tan^{-1}(x) + \tan^{-1} C$$

$$-\tan^{-1}(y) - (-\tan^{-1}(x)) = -\tan^{-1} C$$

$$-\tan^{-1}(y-x) = \tan^{-1} C$$

$$y-x = C$$

$$-\tan^{-1}(y) - \tan^{-1}(x) = C$$

⑧ $(1+x^2)xy \frac{dy}{dx} = (1+y^2)(1+x+x^2)$

$(1+x^2)x$

Homogeneous function :-

A function $f(x, y)$ is said to be homogeneous function if of degree 'n' if $f(kx, ky) = k^n f(x, y)$

Homogeneous differential equation :-

A differential equation of the form

$\frac{dy}{dx} = f(x, y)$ is said to be homogeneous differential equation if the function $f(x, y)$ is a homogeneous function of degree 0.

Working rule :-

The above differential equation can be solved by taking $y = vx$ then

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

Now, the DE can be written as

$$v + x \frac{dv}{dx} = f(x, vx)$$

$$x \frac{dv}{dx} = f(x, vx) - v$$

Using variable - separable form find its solution is of the form $\phi(x, v, c) = 0$

General solution of given differential equation is obtained by substituting

$$v = y/x \text{ that is the General}$$

$$\text{solution is } \phi(x, y/x, c) = 0$$

① solve $(x^2 - y^2) dx - xy dy = 0$

Given that $(x^2 - y^2) dx - xy dy = 0$

$$\frac{dy}{dx} = \frac{x^2 - y^2}{xy} \quad \text{--- ①}$$

put $y = vx$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

— from ① $\Rightarrow$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{x^2 - (vx)^2}{x(vx)}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{x^2(1 - v^2)}{x^2 v}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{1 - v^2}{v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 - v^2}{v} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 - 2v^2}{v}$$

$$\Rightarrow \frac{v}{1 - 2v^2} dv = \frac{x dx}{x}$$

$$\int \frac{v}{1 - 2v^2} dv = \int \frac{1}{x} dx + \log c$$

$$\frac{-1}{4} \int \frac{-4v}{1 - 2v^2} dv = \int \frac{1}{x} dx + \log c$$

$$\frac{-1}{4} \log(1 - 2v^2) = \log x + \log c$$

$$\frac{-1}{4} \log(1 - 2v^2) = \log(xc)$$

$$\log(1-2v^2)^{1/4} = \log(cx)$$

$$(1-2v^2)^{1/4} = 10 \cdot cx$$

$$\text{put } v = \frac{y}{x}$$

$$\left(1-2\left(\frac{y}{x}\right)^2\right)^{1/4} = cx$$

$$\left[\frac{x^2-2y^2}{x^2}\right]^{1/4} = cx$$

$$\left[\frac{x^2}{x^2-2y^2}\right]^{1/4} = cx$$

$$\frac{x^{1/2}}{(x^2-2y^2)^{1/4}} = cx$$

$$C(x^2-2y^2)^{1/4} = \frac{1}{\sqrt{x}}$$

$$\sqrt{x}(x^2-2y^2)^{1/4} = \frac{1}{C}$$

*** Exact Differential Equation :-

Any first order Differential Equation which is obtained by differentiation of its general solution without any elimination or reduction of terms is known as exact differential equation.

If $f(x, y) = C$ is the general solution

Then $df = 0$

$$\Rightarrow \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

$$\text{Let } M(x,y) = \frac{\partial f}{\partial x} \text{ \& } N(x,y) = \frac{\partial f}{\partial y}$$

Then the above equation can be written as

$$M(x,y) dx + N(x,y) dy = 0$$

(or)

$$M dx + N dy = 0 \text{ which represents}$$

also an exact differential equation.

$$\text{Now } \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ \& } \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\therefore \left[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \right]$$

Thus, the necessary condition for a differential equation to be an exact differential equation is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The General solution of exact D.E is

$$\left[\int M(x,y) dx + \int (\text{terms free from } x \text{ in } N) dy = c \right]$$

c is constant

$$\textcircled{1} \text{ Solve } (1 + 2xy \cos x^2 - 2xy) dx + (\sin x^2 - x^2) dy = 0$$

Given that

$$(1 + 2xy \cos x^2 - 2xy) dx + (\sin x^2 - x^2) dy = 0 \text{ --- } \textcircled{1}$$

\textcircled{1} is of the form $M dx + N dy = 0$

$$\text{where } M = 1 + 2xy \cos x^2 - 2xy \text{ \& } N = \sin x^2 - x^2$$

$$\text{Now } \frac{\partial M}{\partial y} = 0 + 2x \cos x^2 - 2x \text{ \& } \frac{\partial N}{\partial x} = \cos x^2 (2x) - 2x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ $\odot$ is an exact D.E

$\therefore$ General solution $\odot$ is

$$\int M dx + \int (\text{terms of } N \text{ not containing } x) dy = C$$

(constant)

$$\rightarrow \int (1 + 2xy \cos x^2 - 2xy) dx + \int 0 dy = C$$

$$\rightarrow x + y \int 2x (\cos x^2) - 2y \int x dx = C$$

— take $x^2 = t$
 $2x dx = dt$

$$\Rightarrow x + y \int \cos t dt - 2y \left(\frac{x^2}{2} \right) = C$$

$$\Rightarrow x + y [\sin t] - x^2 y = C$$

$$\Rightarrow x + y \sin x^2 - x^2 y = C,$$

$\odot$ Solve $(xe^{xy} + 2y) \frac{dy}{dx} + ye^{xy} = 0$

Given that

$$(xe^{xy} + 2y) \frac{dy}{dx} + ye^{xy} = 0 \quad \text{--- } \odot$$

$$(xe^{xy} + 2y) \frac{dy}{dx} = -ye^{xy}$$

$$(xe^{xy} + 2y) dy = -ye^{xy} dx$$

$$(xe^{xy} + 2y) dy + ye^{xy} dx = 0 \quad \text{--- } \odot$$

$\odot$ is in the form of $Mdx + Ndy = 0$

$$M = ye^{xy} \quad \text{and } N = xe^{xy} + 2y$$

$$\frac{\partial M}{\partial y} = ye^{xy} x + e^{xy} = e^{xy}(yx + 1)$$

or $dx + dy$

$$\frac{\partial N}{\partial x} = x e^{xy} y + e^{2y}$$

$$= e^{xy} (xy + 1)$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ (i) is an exact D.E

General solution (i) is

$$\int M dx + \int [\text{term of N not containing } x] dy = C$$

$$\int e^{xy}$$

$$\int y e^{xy} dx + \int 2y dy = C$$

$$y \frac{e^{xy}}{y} + 2 \frac{y^2}{2} = C$$

$$e^{xy} + y^2 = C$$

(3) solve $2xydy - (x^2 - y^2 + 1)dx = 0$

Given that

$$2xydy - (x^2 - y^2 + 1)dx = 0 \quad \text{--- (1)}$$

(i) is in the form of $Mdx + Ndy = 0$

$$M = -(x^2 - y^2 + 1) \quad N = 2xy$$

$$\frac{\partial M}{\partial y} = 0 + 2y$$

$$\frac{\partial N}{\partial x} = 2y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow$ is an exact D.E

General solution (3) is

$$\int M dx + \int (\text{net containing } x \text{ terms } P(x)) dy = 0$$

$$\int (-x^2 + y^2 - 1) dx + \int 0 dy = C$$

$$\int -\frac{x^2}{2} + \frac{y^2}{2} - x = C$$

$$-\frac{x^3}{3} + \frac{y^3}{3} - x = C$$

$$-\frac{x^3}{3} + y \int 1 dx - \int 1 dx = C$$

$$-\frac{x^3}{3} + yx - x = C$$

$$-x^3 + 3y^2x - 3x = C$$

$$-x^3 + 3x(y^2 - 1) = C$$

$$3x(y^2 - 1) + x^3$$

④ Solve $(\cos x + y \sin x) dx - \cos x dy = 0$

Given that

$$(\cos x + y \sin x) dx - \cos x dy = 0 \quad \text{--- (1)}$$

① P_0, P_1 the form $M dx + N dy = 0$

where

$$M = \cos x + y \sin x \quad \text{and } N = -\cos x$$

$$\frac{\partial M}{\partial y} = 0 + \sin x$$

$$\frac{\partial N}{\partial x} = -\sin x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

① Is an exact D.E
General solution ① is

$$\int M dx + \int \text{C-terms of N not containing x} dy = C$$

$$\int \cos x + y \sin x dx + \int 0 dy = C$$

$$\int \sin x + y \cos x = C$$

$$\sin x + y \cos x = C$$

⑤. $(x+y-1)dy - (x-y+2)dx = 0$.

Given that

$$(x+y-1)dy - (x-y+2)dx = ①$$

① Is in the form of $Mdx + Ndy = 0$

where

$$M = -(x-y+2) \quad \text{if} \quad N = (x+y-1)$$
$$-x+y+2 \quad \quad \quad x+y-1$$

$$\frac{\partial M}{\partial y} = 0+1+0 \quad \quad \quad \frac{\partial N}{\partial x} = 1+0-0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

① Is an exact D.E

General solution ① is

$$\int M dx + \int \text{C-terms of N not containing x} dy = C$$

$$\int -x+y+2 dx + \int y-1 dy = C$$

$$\rightarrow \int -\frac{x^2}{2} + 2x + \frac{y^2}{2} - y = C$$

$$-x^2 + 4x + y^2 - 2y = C$$

⑤ Variable - Separable method.

①. solve $\frac{dy}{dx} + \sqrt{\frac{1+y^2}{1+x^2}} = 0$

$$\frac{dy}{dx} = -\sqrt{\frac{1+y^2}{1+x^2}}$$

$$\frac{dy}{dx} = \frac{\sqrt{1+y^2}}{\sqrt{1+x^2}}$$

$$\frac{1}{\sqrt{1+y^2}} dy = \frac{-1}{\sqrt{1+x^2}} dx$$

$$\frac{1}{\sqrt{1+y^2}} dy + \frac{1}{\sqrt{1+x^2}} dx = 0$$

on integrating.

$$\int \frac{1}{\sqrt{1+y^2}} dy + \int \frac{1}{\sqrt{1+x^2}} dx = C$$

$$\sinh^{-1} y + \sinh^{-1} x = C$$

②. $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$

③. $(y+y^2)dx + xy dy = 0$

$$(y+y^2)dx = -xy dy$$

$$\frac{1}{x} dx = \frac{-y}{y+y^2} dy$$

$$\frac{1}{x} dx + \frac{y}{y+y^2} dy = 0$$

$$\int \frac{1}{x} dx + \int \frac{y}{y+y^2} dy = C$$

② solve $\frac{dy}{dx} = \frac{-x}{y}$ where $x=1, y=\sqrt{3}$

$$y dy = -x dx$$

$$\int y dy = -\int x dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C$$

$$x^2 + y^2 = C$$

$$1^2 + (\sqrt{3})^2 = C$$

$$4 = C$$

$$x^2 + y^2 = 4$$

③ $\frac{dy}{dx} = (4x + y + 1)^2$

$$4x + y + 1 = z$$

$$4 + \frac{dy}{dz} = \frac{dz}{dx}$$

$$\frac{dz}{dx} - 4 = (z)^2$$

$$\frac{dz}{dz} = 4 + z^2$$

$$\frac{dz}{4 + z^2} = dx$$

$$\int \frac{dz}{4 + z^2} = \int dx + C$$

$$\frac{1}{2} \tan^{-1} \frac{z}{2} = x + C$$

$$\frac{1}{2} \tan^{-1} \left[\frac{4x + y + 1}{2} \right] = x + C$$

$$(4) (x^2 - yx^2) \frac{dy}{dx} + (y^2 + x^2y^2) = 0.$$

$$(x^2 - yx^2) \frac{dy}{dx} = - (y^2 + x^2y^2)$$

$$x^2 - yx^2 dy = -y^2 - x^2y^2 dx$$

$$x^2(1-y) dy = -y^2(1+x^2) dx$$

$$\frac{(1-y) dy}{y^2} = \frac{1+x^2}{x^2} dx$$

$$\int \frac{1-y}{y^2} dy = \int \frac{1+x^2}{x^2} dx$$

$$\int \frac{1}{y^2} dy - \int \frac{1}{y} dy = \int \frac{1}{x^2} + \int dx - C$$

$$\frac{1}{y} - \log y = \frac{1}{x} + x + C,$$

$$(5) (e^y + 1) \cos x dx + e^y \sin x dy = 0.$$

$$(e^y + 1) \cos x dx = -e^y \sin x dy$$

$$\frac{\cos x}{\sin x} dx = \frac{-e^y}{e^y + 1} dy$$

$$\frac{\cos x}{\sin x} dx + \frac{e^y}{e^y + 1} dy = 0$$

$$\int \frac{\cos x}{\sin x} dx + \int \frac{e^y}{e^y + 1} dy = C$$

$$\int \cot x dx + \int \frac{e^y}{e^y + 1} dy = C$$

$$\log |\sin x| + \log |e^y + 1| = \log C$$

$$(\sin x)(e^y + 1) = C$$

$$(6) \quad (y + yx) dx + (x + xy) dy = 0$$

$$y(1+x) dx + x(1+y) dy = 0$$

$$y(1+x) dx = -x(1+y) dy$$

$$\frac{1+x}{x} dx = -\frac{1+y}{y} dy$$

$$\frac{1+x}{x} dx + \frac{1+y}{y} dy = 0$$

$$\int \frac{1+x}{x} dx + \int \frac{1+y}{y} dy = C$$

$$\int \frac{1}{x} + \int dx + \int \frac{1}{y} dy + \int dy = C$$

$$\log x + x + \log y + y = C$$

$$\log xy + x + y = C$$

$$(7) \quad x \frac{dy}{dx} + \cot y = 0$$

$$x \frac{dy}{dx} = -\cot y$$

$$\left(\frac{x}{dx} = -\frac{\cot y}{dy} \right)$$

$$\frac{dy}{dx} = -\cot y + \frac{1}{x}$$

$$\int \cot y \, dy = -\frac{1}{x} \, dx$$

$$\int \cot y \, dy + \int \frac{1}{x} \, dx = \log C$$

$$\log(\sec y) + \log x = \log C$$

$$\log(x \sec y) = \log C$$

$$x(\sec y) = C$$

$$(8) \quad \frac{dy}{dx} = \frac{x+x^2}{y-y^2}$$

$$(y-y^2) \, dy = (x+x^2) \, dx$$

$$\int y \, dy - \int y^2 \, dy = \int x \, dx + \int x^2 \, dx$$

$$\frac{y^2}{2} - \frac{y^3}{3} = \frac{x^2}{2} + \frac{x^3}{3} + C$$

$$(9) \quad 3e^x \tan y \, dx + (1-e^x) \sec^2 y \, dy = 0$$

$$3e^x \tan y \, dx = -(1-e^x) \sec^2 y \, dy$$

$$\frac{3e^x}{1-e^x} \, dx = -\frac{\sec^2 y}{\tan y} \, dy$$

$$\frac{3e^x}{1-e^x} \, dx + \frac{\sec^2 y}{\tan y} \, dy = 0$$

$$\int \frac{2e^x}{1-e^x} dx + \int \frac{2e^y}{1-\tan y} dy = c$$

$$-2 \log|1-e^x| + \log|1-\tan y| = c$$

$$\log \left| \frac{1-\tan y}{(1-e^x)^2} \right| = c$$

(10) $\cos \theta dr - r \sin \theta d\theta = 0$

$$\cos \theta dr = r \sin \theta d\theta$$

$$\frac{1}{r} dr = \frac{\sin \theta}{\cos \theta} d\theta$$

$$\frac{1}{r} dr = \tan \theta d\theta$$

$$\int \frac{1}{r} dr = \int \tan \theta d\theta = c$$

$$\log r - \log (\cos \theta) = \log c$$

$$\log r \cos \theta = \log c$$

$$r \cos \theta = c$$

(Ex 1) Exact differential equations

(2) Solve $\left[y\left(1+\frac{1}{x}\right) + \cos y\right] dx + (x + \log x - x \sin y) dy = 0$

Sol

$$\left[y\left(1+\frac{1}{x}\right) + \cos y\right] dx + (x + \log x - x \sin y) dy = 0 \quad \text{--- (1)}$$

(1) $P_1 dx + P_2 dy = 0$

$$M = y\left(1+\frac{1}{x}\right) + \cos y$$

$$N = x + \log x - x \sin y$$

$$\frac{\partial M}{\partial y} = y + \frac{y}{x} + \cos y$$

$$= 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

① is an exact D.E

General solution ①.

$$\int M dx + \int N (\text{terms of } N \text{ not containing } x) dy$$

$$\int y \left(1 + \frac{1}{x}\right) + \cos y + \int 0 dy = C$$

$$y [x + \log x] + \cos y + 0 = C$$

$$xy + \log yx + x \cos y = C,$$

② $(x^2 - ay) dx = (ax - y^2) dy$

Given that

$$(x^2 - ay) dx = (ax - y^2) dy \text{ is } \textcircled{1}.$$

①. is of the form $M dx + N dy = 0$

$$(x^2 - ay) dx - (ax - y^2) dy = 0,$$

$$M = (x^2 - ay) \quad \& \quad N = -a + y^2$$

$$\frac{\partial M}{\partial y} = -a \quad \frac{\partial N}{\partial x} = -a$$

$$\frac{\partial N}{\partial x} = -a$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

where equation (1) is exact

General solution of (1)

$$\int M dx + \int C \text{ not containing } x \text{ terms in } N dy = C$$

$y \text{ is constant}$

$$\int (x^2 - ay) dx + \int y^2 dy = C$$

$$\int \frac{x^3}{3} - axy + \frac{y^3}{3} = C$$

$$x^3 - 3axy + y^3 = 3C$$

$$x^3 - 3axy + y^3 = k$$

(4)

Given that

$$ye^{xy} dx + (xe^{xy} + 2y) dy = 0$$

Given that

$$ye^{xy} dx + (xe^{xy} + 2y) dy = 0 \quad \text{--- (1)}$$

(1) is in the form $Mdx + Ndy = 0$

where $M = ye^{xy}$ $N = xe^{xy} + 2y$

Now

$$\frac{\partial M}{\partial y} = e^{xy} + y e^{xy}(x) \\ = e^{xy}(1+xy)$$

$$\frac{\partial N}{\partial x} = e^{xy} + x e^{xy}(y) \\ = e^{xy}(1+xy)$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

where equation ① is exact

The General Solution

$$\int M dx + \int (N \text{ not containing } x\text{-terms}) dy = C$$

$$\int y e^{xy} dx + \int 2y dy = C$$

$$y \frac{e^{xy}}{y} + \frac{2y^2}{2} = C$$

$$e^{xy} + y^2 = C$$

$$\textcircled{5} \quad \frac{3x}{y^3} dx + \frac{y^3 - 3x^2}{y^4} dy = 0$$

Given that

$$\frac{3x}{y^3} dx + \frac{y^3 - 3x^2}{y^4} dy = 0 \quad \text{--- ①}$$

$$\textcircled{1} \quad P_0 \quad P_1 \quad dx \quad dM + dN \\ x M dx + N dy = 0$$

$$M = \frac{3x}{y^3} \quad N = \frac{y^3 - 3x^2}{y^4} \\ \frac{1}{y^2} - \frac{3x^2}{y^4}$$

$$\frac{\partial M}{\partial y} = 2x \left[\frac{-3}{y^4} \right]$$

$$\frac{\partial N}{\partial x} = 0 - \frac{6x}{y^4}$$

$$= \frac{-6x}{y^4}$$

$$= \frac{-6x}{y^4}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

where equation ① is exact

The General solution

$$\int M dx + \int (N \text{ not containing } x \text{ terms}) dy = C$$

$$\int \frac{2x}{y^3} dx + \int \frac{1}{y^2} dy = C$$

$$\int \frac{1}{y^3} \cdot \frac{x^2}{2} + \frac{-1}{y} = C$$

$$\frac{x^2}{2y^3} - \frac{1}{y} = C$$

$$x^2 - y^2 = y^3 C$$

$$x^2 - y^2 = Cy^3$$

$$\textcircled{E} \quad y \sin 2x dx - (1 + y^2 + \cos^2 x) dy = 0$$

Given that

$$y \sin 2x dx - (1 + y^2 + \cos^2 x) dy = 0 \quad \text{--- ①}$$

is in the form of $M dx + N dy = 0$.

$$M = y \sin 2x$$

$$N = -1 - y^2 - \cos^2 x$$

$$\frac{\partial M}{\partial y} = \sin 2x$$

$$\begin{aligned}\frac{\partial N}{\partial x} &= 0 - 0 - \sin^2 x \\ &\quad 2 \cos x (-\sin x) \\ &\quad + 2 \cos x \sin x \\ &= \sin 2x\end{aligned}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

where equation (1) is exact

General solution

$$\int M dx + \int (N \text{ not containing } x) dy = C$$

$$\int y \sin 2x + \int (4y^3) dy = C$$

$$\int y \left[-\frac{\cos 2x}{2} \right] + \left[y - \frac{y^3}{3} \right] = C$$

$$\frac{y}{2} \cos 2x + y - \frac{y^3}{3} = C$$

$$3y \cos 2x + 6y - 2y^3 = 6C$$

$$3y \cos 2x + 6y - 2y^3 = K$$

⑦ $(\sec x \tan x \tan y - e^x) dx + \sec x \sec^2 y dy = 0$

Given that

$$(\sec x \tan x \tan y - e^x) dx + \sec x \sec^2 y dy = 0 \quad \text{--- (1)}$$

① & ② in the form of $M dx + N dy = 0$

$$M = \sec x \tan x \tan y - e^x \text{ if } N = \sec x \sec^2 y$$

$$\frac{\partial M}{\partial y} = \sec x \tan x \sec^2 y - 0 \quad \frac{\partial N}{\partial x} = \tan y \sec^2 y$$

$$= \sec x \tan x \sec^2 y$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

where equation (1) is exact

$$\int M dx + \int (\text{where } n \text{ not containing } x) dy = C$$

$$\int (\sec x \tan x \tan y - e^x) dx + \int \sec^2 y dy = C$$

$$\Rightarrow \tan y (\sec x) - e^x = C$$

$$\tan y (\sec x) - e^x = C$$

Equation Reducible to exact is

$\Rightarrow$ If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then the differential equation $M dx + N dy = 0$ is called not an exact differential equation.

Some times if the given differential equation is not a exact P.E. can be made an exact D.E. by multiplying it with a suitable function or factor ($\mu(x, y) \neq 0$). In this case this factor is called an Integrating factor.

Methods to find

Methods to find an Integrating factor.

Method 1: If $Mdx + Ndy = 0$ is a homogeneous Differential Equation then $\frac{1}{Mx + Ny}$ is an Integrating factor $Mdx + Ndy = 0$

① Solve the D.E $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$
Given that

$$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0 \quad \text{--- ①}$$

① is in the form of $Mdx + Ndy = 0$

where

$$M = x^2y - 2xy^2 \quad \text{and} \quad N = -x^3 + 3x^2y$$

$$\frac{\partial M}{\partial y} = x^2 - 2x$$

$$\frac{\partial N}{\partial x} = -3x^2 + 6xy$$

$$\left[\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \right]$$

where ① is not exact D.E & is homogeneous

$$\text{Integrating factor} = \frac{1}{Mx + Ny}$$

$$\begin{aligned} Mx + Ny &= (x^2y - 2xy^2)x + (-x^3 + 3x^2y)y \\ &= x^3y - 2x^2y^2 - x^3 + 3x^2y^2 \end{aligned}$$

$$= x^2y^2 \neq 0$$

$$\text{Integrating factor} = x^2y^2$$

$$M_1 = x^2 y^2 (x^2 y - 2xy^2) \quad \& \quad N_1 = x^2 y^2 (-x^3 + 3x^2 y)$$

$$= x^4 y^3 - 2x^3 y^4 \quad \& \quad N_1 = -x^5 y^2 + 3x^4 y^3$$

$$\frac{\partial M_1}{\partial y} = x^4 y^3 - 2x^3 y^4$$

$$= 4x^3 y^2 - 2 \cdot 3(x^3)4(y^3) \quad - 5x^4$$

$$= 12x^3 y^2 - 24x^3 y^3$$

$$\frac{\partial N_1}{\partial x} = -x^5 y^2 + 3x^4 y^3$$

$$= -5x^4 y^2 + 12x^3 y^3$$

Now multiply ① with $\frac{1}{x^2 y^2}$

$$\frac{x^2 y - 2xy^2}{x^2 y^2} dx - \frac{x^3 - 3x^2 y}{x^2 y^2} dy = 0.$$

$$\left[\frac{1}{y} - \frac{2}{x} \right] dx - \left[\frac{x}{y^2} - \frac{3}{y} \right] dy = 0. \quad \text{--- ②}$$

$M_1 =$ Equation ② is of the form $M_1 dx + N_1 dy = 0$

where $M_1 = \frac{1}{y} - \frac{2}{x}$ $N_1 = -\frac{x}{y^2} + \frac{3}{y}$

$$\frac{\partial M_1}{\partial y} = -\frac{1}{y^2} - 0.$$

$$\frac{\partial N_1}{\partial x} = -\frac{1}{y^2} - 0.$$

$$\left[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \right]$$

where equation ② is exact

$$\int M_1 dx + \int (\text{free of } x \text{ in } N_1) dy = c$$

$$\int \left(\frac{1}{y} - \frac{2}{x} \right)^{\frac{1}{2}} \int \frac{3}{y} dy = C$$

$$\frac{x}{y} - 2 \log x + 3 \log y = C$$

$$\frac{x}{y} - \log x^2 + \log y^3 = C$$

$$\frac{x}{y} + \log \left(\frac{y^3}{x^2} \right) = C \quad \text{or} \quad \frac{x}{y} - \log \left(\frac{x^2}{y^3} \right) = C$$

$$\textcircled{2} \quad y(x+y)dx - x(y-x)dy = 0.$$

Given that

$$y(x+y)dx - x(y-x)dy = 0 \quad \text{--- (1)}$$

equation (1) is in the form of

$$Mdx + Ndy = 0,$$

where

$$M = yx + y^2$$

$$N = -xy + x^2$$

$$\frac{\partial M}{\partial y} = x + 2y$$

$$\frac{\partial N}{\partial x} = -y + 2x$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

(1) is not an exact DE

$$\text{Integrating factor} = \frac{1}{Mx + Ny}$$

$$\begin{aligned} Mx + Ny &= (yx + y^2)x + (-xy + x^2)y \\ &= yx^2 + xy^2 + (-xy^2 + x^2y) = yx^2 + x^2y \end{aligned}$$

$$= yx^2 + 2y^2 - y^2 + 2xy$$

$$= yx^2 + \cancel{2y^2} - \cancel{y^2} + 2y$$

$$= 3xy$$

Integrating factor = $3x^2y$

Now multiply (1) with $\frac{1}{3x^2y}$.

$$\frac{yx+y^2}{2x^2y} dx - \frac{xy-x^2}{3x^2y} dy = 0$$

$$\frac{1}{2x} + \frac{y}{2x^2} dx - \left[\frac{1}{3x} - \frac{1}{2y} \right] = 0$$

$$M_1 = \frac{1}{2x} + \frac{y}{2x^2}, \quad N_1 = -\frac{1}{3x} + \frac{1}{2y}$$

$$\frac{\partial M_1}{\partial y} = 0 + \frac{1}{2x^2}, \quad \frac{\partial N_1}{\partial x} = -\frac{1}{3} \left[\frac{-1}{x^2} \right] + 0$$

$$= \frac{1}{3x^2}$$

$$\boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$$

② P is an exact

then General Solution

$$\int \frac{1}{2x} + \frac{y}{2x^2} dx + \int \frac{1}{2y} dy = C$$

$$\frac{1}{2} \log x + \frac{y}{2} \left[-\frac{1}{x} \right] + \frac{1}{2} \log y = C$$

$$\frac{1}{2} \log x - \frac{y}{2x} + \frac{1}{2} \log y = C$$

$$3xy^2 + 2x^2y = 0$$

$$y(1) = 1$$

Given that

$$3xy^2 + 2x^2y = 0 \quad \text{--- (1)}$$

It is in the form of $Mdx + Ndy = 0$

where

$$M = 3xy$$

$$N = 2x$$

$$\frac{\partial M}{\partial y} = 3$$

$$\frac{\partial N}{\partial x} = 2$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

equation (1) is not in exact.

$$\text{Integrating factor} = \frac{1}{Mx + Ny}$$

$$\begin{aligned} Mx + Ny &= (3y)x + (2x)y \\ &= 3xy + 2xy \end{aligned}$$

$$= 5xy$$

$$\text{Integrating factor} = 5xy$$

Now Multiply (1) with $\frac{1}{5xy}$

$$\frac{3y}{5xy} dx + \frac{2x}{5xy} dy = 0$$

$$\frac{3}{5x} dx + \frac{2}{5y} dy = 0 \quad \text{--- (2)}$$

It is in the form of $Mdx + Ndy = 0$

$$M_1 = \frac{3}{5x}$$

$$N_1 = \frac{2}{5y}$$

$$\frac{\partial M_1}{\partial y} = 0$$

$$\frac{\partial N_1}{\partial x} = 0$$

$$\left[\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \right]$$

equation (2) is not an exact
Than General solution

$$\int M dx + \int (\text{free of } x \text{ Pn } N) dy = C$$

$$\int \frac{3}{5x} dx + \int \frac{2}{5y} dy = C$$

$$\frac{3}{5} \log x + \frac{2}{5} \log y = C$$

$$\frac{1}{5} [3 \log x + 2 \log y] = C$$

$$3 \log x + 2 \log y = 5C$$

$$\log x^3 + \log y^2 = 5C$$

$$\log (x^3 y^2) = 5 \log C$$

$$(\log (x^3 y^2) = k_{11})$$

$$x^3 y^2 = C^5 \text{ --- (3)}$$

(4)

$$x^2 y dx - (x^3 - y^3) dy = 0$$

Given that

$$x^2 y dx - (x^3 - y^3) dy = 0 \text{ --- (1)}$$

(1) is in the form of $M dx + N dy = 0$

where

$$M = x^2 y$$

$$N = -x^3 + y^3$$

$$\text{G.P. } y(1) = 1$$

$$\text{put } y=1 \text{ when } x=1$$

$$\text{put } y=1, x=1 \text{ in (3)}$$

$$1 \times 1 = C^5 \rightarrow C=1$$

$$\text{put } C=1 \text{ in eqn (3)}$$

$$\text{then particular solution } x^3 y^2 = 1$$

$$\frac{\partial M}{\partial y} = x^2 y$$

$$= x^2$$

$$\frac{\partial N}{\partial x} = -x^3 + y^3$$

$$= -2x^2$$

$$\left[\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \right]$$

equation (1) is not exact

$$\text{Integrating factor} = \frac{1}{Mx + Ny}$$

$$\begin{aligned} Mx + Ny &= (x^2 y)x + (-x^3 + y^3)y \\ &= x^3 y - x^3 y + y^4 \\ &= y^4 \end{aligned}$$

$$\text{Integrating factor} = \frac{1}{y^4}$$

$$\frac{x^2 y}{y^4} dx - \frac{(x^3 - y^3)}{y^4} dy = 0$$

$$\frac{x^2}{y^3} dx - \left(\frac{x^3}{y^4} + \frac{1}{y^4} \right) dy = 0 \quad \text{--- (2)}$$

$$M_1 = \frac{x^2}{y^3}$$

$$N_1 = -\frac{x^3}{y^4} + \frac{1}{y^4}$$

$$\frac{\partial M_1}{\partial y} = x^2 \left[\frac{1}{y^3} \right]$$

$$\begin{aligned} \frac{\partial N_1}{\partial x} &= -\frac{x^3}{y^4} + \frac{1}{y^4} \\ &= -1 \end{aligned}$$

$$= x^2 y^3$$

$$= -3x^2 y^4$$

$$= -3x^2 y^4$$

$$= -3x^2 y^4$$

$$\left[\frac{\partial M_1}{\partial y} = \frac{\partial M_2}{\partial x} \right]$$

equation (2) is exact

$$\text{Integrating factor} = \frac{1}{Mx + Ny}$$

General solution

$$\int M dx + \int (\text{free of } x \text{ in } N) dy = C$$

$$\int \frac{x^2}{y^3} dx + \int \left(-\frac{x^3}{y^4} + \frac{1}{y} \right) dy = C$$

$$\int \frac{1}{y^3} x^2 dx + \int \frac{1}{y} dy = C$$

$$\int \frac{1}{y^3} x^2 dx + \int \frac{1}{y} dy = C$$

$$\frac{1}{y^3} \frac{x^2+1}{2+1} + \log y = C$$

$$\frac{1}{y^3} \frac{x^3}{3} + \log y = C$$

$$\frac{1}{3} \frac{x^3}{y^3} + \log y = C$$

$$(5) \quad \frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$$

$$xy^2 dy = x^3 + y^3 dx$$

$$(x^3 + y^3)dx - xy^2 dy = 0 \quad \text{--- (1)}$$

(1) is in the form of $Mdx + Ndy = 0$

$$M = x^3 + y^3$$

$$N = -xy^2$$

$$\frac{\partial M}{\partial y} = 0 + 3y^2$$

$$\frac{\partial N}{\partial x} = -y^2$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

(1) is not in form of exact

$$\text{Integrating factor} = \frac{1}{Mx + Ny}$$

$$\begin{aligned} Mx + Ny &= (x^3 + y^3) \cdot x + (-xy^2)y \\ &= x^4 + \cancel{xy^3} - \cancel{xy^3} \\ &= x^4 \end{aligned}$$

$$\text{Integrating factor} = \frac{1}{Mx + Ny}$$

$$\frac{x^3+y^3}{x^4} dx - \frac{xy^2}{x^4} dy = 0$$

$$\frac{1}{x} + \frac{y^3}{x^4} dx - \frac{y^2}{x^3} dy = 0 \quad \text{--- (2)}$$

$$M_1 = \frac{1}{x} + \frac{y^3}{x^4}$$

$$N_1 = \frac{-y^3}{x^3}$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{x^4} [y^3]$$

$$= \frac{1}{x^4} 3y^2$$

$$= 3y^2 x^{-4}$$

$$\frac{\partial N_1}{\partial x} = -y^3 \left[\frac{1}{x^3} \right]$$

$$= -y^3 [-x^{-3}]$$

$$= x^3 y^3$$

$$= 3y^3 x^{-4}$$

$$= 3y^3 x^{-4}$$

$$\boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$$

equation ② is in the form of exact
Then General solution

$$\int M dx + \int (\text{free of } x \text{ in } N) dy = C$$

$$\int \frac{1}{x} + \frac{y^3}{x^4} + \int 0 dy = C$$

$$\log x + y^3 [x^{-4}] = C$$

$$\log x + y^3 \frac{x^{-4+1}}{-4+1} = C$$

$$\log x + y^3 \frac{x^{-3}}{-3} = C,$$

$$\log x - \frac{y^3}{3x^3} = C,$$

$$⑥ \quad 2xy \, dx + (y^2 - x^2) \, dy = 0.$$

Given that

$$2xy \, dx + (y^2 - x^2) \, dy = 0 \quad \text{--- (1)}$$

① is in the form of $M \, dx + N \, dy = 0$
where

$$M = 2xy \qquad N = y^2 - x^2$$

$$\frac{\partial M}{\partial y} = 2x$$

$$\frac{\partial N}{\partial x} = 0 - 2x$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

① is not an exact

$$\text{Integrating factor} = \frac{1}{Mx + Ny}$$

$$Mx + Ny = (2xy)x + (y^2 - x^2)y$$

$$= 2x^2y + y^3 - x^2y$$

$$= x^2y + y^3$$

$$\text{Integrating factor} = \frac{1}{x^2y + y^3}$$

$$\frac{1}{x^2y + y^3} (2xy) \, dx + \frac{y^2 - x^2}{x^2y + y^3} \, dy = 0$$

$$\frac{2xy}{x^2y + y^3} \, dx + \frac{y^2}{x^2y + y^3} - \frac{x^2}{x^2y + y^3} \, dy = 0$$

$$\frac{\partial L}{\partial^2 + y^2} dx + \frac{y}{x^2 + y^2} - \frac{x^2}{x^2 y + y^3} dy = 0.$$

$$\frac{\partial L}{\partial^2 + y^2} dx + \frac{(y^2 - x^2)}{x^2 y + y^3} dy = 0.$$

$$M_1 = \frac{\partial L}{\partial^2 + y^2} \quad N_1 = \frac{y^2 - x^2}{x^2 y + y^3}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial}{\partial y} \left[\frac{-1}{(x^2 + y^2)^2} \right] \quad \frac{\partial N_1}{\partial x} = \frac{(x^2 + y^3)(-2x) - (y^2 - x^2)(2xy)}{(x^2 y + y^3)^2}$$

$$= \frac{-4xy}{(x^2 + y^2)^2} \quad \frac{(x^2 y + y^3)(-2x) - (y^2 - x^2)(2xy)}{(x^2 y + y^3)^2}$$

$$\Rightarrow \frac{-2xy^2 + 2xy^3 - 2xy^3 + 2xy^2}{(x^2 y + y^3)^2}$$

$$= \frac{-4xy^2}{y^2(x^2 + y^2)^2}$$

$$= \frac{-4xy}{(x^2 + y^2)^2}$$

$$\left[\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \right]$$

eqn ② is exact.

General solution

$$\int M dx + \int (\text{free of } x \text{ in } N) dy = c$$

$$(7) \quad xy dx - (x^2 + 2y^2) dy = 0$$

$$(8) \quad (8xy^2 - y^3) dx - (2x^2y - xy^2) dy = 0$$

$$\int \frac{dx}{x^2 + y^2} + \int 0 dy = C$$

$$\log(x^2 + y^2) = \log C$$

$$x^2 + y^2 = C$$

$$(9) \quad xy dx - (x^2 + 2y^2) dy = 0$$

Given that $xy dx -$

Method II :- If $Mdx + Ndy = 0$ is of the form $y f_1(xy) dx + x f_2(xy) dy = 0$

Then $IF = \frac{1}{Mx - Ny}$, $Mx - Ny \neq 0$

$$(10) \quad \text{Solve } (1+xy)y dx + (1-xy)x dy = 0$$

$$\text{Given } (1+xy)y dx + (1-xy)x dy = 0 \quad (1)$$

(1) is of the form $Mdx + Ndy = 0$

where

$$M = (1+xy)y \\ y + xy^2$$

$$N = (1-xy)x \\ = x - x^2y$$

$$\frac{\partial M}{\partial y} = 1 + 2xy$$

$$\frac{\partial N}{\partial x} = 1 - 2xy$$

where

$$\left[\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \right]$$

eqⁿ ① is not exact is of the form $y - 1, (xy)^{1/2} + \ln(xy) + y^{-1}$

Integrating factor = $\frac{1}{Mx - Ny}$

$$Mx - Ny = (y + xy^2)x - (x - x^2y)y$$

$$= xy + x^2y^2 - xy + x^2y^2$$

$$= 2x^2y^2$$

Integrating factor = $\frac{1}{2x^2y^2}$

$$\frac{(1+xy)y}{2x^2y^2} dx + \frac{(1-xy)x}{2x^2y^2} dy = 0$$

$$\frac{(1+xy)}{2xy} dx + \frac{(1-xy)}{2xy^2} dy = 0$$

$$M_1 = \frac{1+xy}{2x^2y}$$

$$N_1 = \frac{1-xy}{2xy^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{1+xy}{2x^2y} = \frac{1}{2x} \left[\frac{1}{y} + x \right]$$

$$\frac{\partial N_1}{\partial x} = \frac{1}{2y} \left[-\frac{1}{y^2} \right]$$

$$= \frac{1}{2x} \left[\frac{y(x) - (1+xy)(1)}{y^2} \right]$$

$$N_1 = \frac{1 - xy}{2xyL}$$

$$= \frac{1}{2yL} \left[\frac{1}{x} - y \right]$$

$$\frac{\partial N_1}{\partial x} = \frac{1}{2yL} \left(-\frac{1}{x^2} \right) = -\frac{1}{2x^2 y L}$$

$$\boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$$

eqn ② is exact

$$\int M dx + \int (\text{free of } x \text{ in } N) dy = C$$

$$\int \frac{1+xy}{2x^2 y} dx + \int \frac{1}{2y} dy = C$$

$$\int \frac{1}{2y} \left[\frac{1}{x^2} \right]$$

$$\int \frac{1}{2x^2 y} + \frac{xy}{2x^2 y} = C$$

$$\int \frac{1}{2y} \left[\frac{x^{-2+1}}{-2+1} \right] + \frac{1}{2} \log x + \frac{d}{2}$$

$$\frac{1}{2y} \left[-\frac{1}{x} \right] + \frac{1}{2} \log x - \frac{1}{2} \log y = C$$

$$\frac{1}{2} \left[-\frac{1}{xy} + \log x - \log y \right] = C$$

$$\frac{-1}{xy} + \log x - \log y = 2C$$

$$-\frac{1}{xy} - \log \left(\frac{y}{x} \right) = 2C$$

$$(3) (x^2y^2 + xy + 1)y dx + (x^2y^2 - xy + 1)x dy = 0$$

Given that

$$(x^2y^2 + xy + 1)y dx + (x^2y^2 - xy + 1)x dy = 0 \quad \text{--- (1)}$$

(1) is in the form of $Mdx + Ndy = 0$

where

$$M = (x^2y^2 + xy + 1)y$$

$$= x^2y^3 + xy^2 + y$$

$$N = (x^2y^2 - xy + 1)x$$

$$= x^3y^2 - xy^2 + x$$

$$\frac{\partial M}{\partial y} = 3x^2y^2 + 2xy + 1$$

$$= \frac{\partial N}{\partial x} = 3x^2y^2 - 2xy + 1$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

eqⁿ (1) is not exact

$$\text{Integrating factor} = \frac{1}{Mx - Ny}$$

$$Mx - Ny = (x^2y^3 + xy^2 + y)x - (x^3y^2 - xy^2 + x)y$$

$$= x^3y^3 + x^2y^2 + xy^2 - x^3y^2 - x^2y^2 - xy^2$$

$$= -x^2y^2$$

$$\text{Integrating factor} = \frac{1}{-x^2y^2}$$

$$\frac{x^2y^3 + xy^2 + y}{-x^2y^2} dx + \frac{x^3y^2 - x^2y^2 + x}{-x^2y^2} dy = 0$$

$$\rightarrow \left[\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y} \right] dx + \left[\frac{x}{2} - \frac{1}{2y} + \frac{1}{2xy^2} \right] dy \quad (2)$$

equation

$$M_1 = \frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}$$

$$N_1 = \frac{x}{2} - \frac{1}{2y} + \frac{1}{2xy^2}$$

$$\left(\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \right)$$

eqn (2) is exact

then

General solution

$$\int M dx + \int (\text{free of } x \text{ in } N) dy = C$$

$$\int \frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y} dx + \int -\frac{1}{2y} dy = C$$

$$\frac{y}{2} \int dx + \frac{1}{2} \int \frac{1}{x} dx + \frac{1}{2y} \int \frac{1}{x^2} dx$$

$$\frac{y}{2}(x) + \frac{1}{2} \log x + \frac{1}{2y} \left[-\frac{1}{x} \right] + \frac{1}{2} \int \frac{1}{y} dy = C$$

$$\frac{xy}{2} + \frac{1}{2} \log x + \frac{1}{2xy} - \frac{1}{2} \log y = C$$

$$\frac{1}{2} \left[xy + \log x - \frac{1}{xy} - \log y \right] = C$$

$$xy + \log x - \frac{1}{xy} - \log y = 2C$$

$$xy - \frac{1}{xy} + \log x - \log y = \partial C$$

$$xy - \frac{1}{xy} + \log\left(\frac{x}{y}\right) = \partial C$$

$$xy - \frac{1}{xy} + \log\left(\frac{x}{y}\right) = k,$$

③ Solve $(2xy+1)y dx + (1+2xy-x^3y^3)x dy = 0$.
Given that

$$(2xy+1)y dx + (1+2xy-x^3y^3)x dy = 0 \quad \text{--- (1)}$$

① Is in the form of $Mdx + Ndy = 0$
where

$$M = (2xy+1)y$$

$$2xy^2 + y$$

$$N = (1+2xy-x^3y^3)x$$

$$x + 2x^2y - x^4y^3$$

$$\frac{\partial M}{\partial y} = 4xy + 1$$

$$\frac{\partial N}{\partial x} = 1 + 4xy - 4x^3y^3$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Is not exact

$$\text{Integrating factor} = \frac{1}{Mx - Ny}$$

$$Mx - Ny = (2xy^2 + y)x - (x + 2x^2y - x^4y^3)y$$

$$2x^2y^2 + xy - xy - 2x^3y^2 + x^4y^4$$

$$= x^4y^4$$

$$\frac{8xy^2 + 9}{x^4y^4} dx +$$

$$\frac{x + 8xy^2 - x^4y^3}{x^4y^4} dy = 0$$

$$\frac{y(2xy+1)}{x^4y^3} dx +$$

$$\frac{x(1+2xy-x^3y^3)}{x^4y^4} dy = 0$$

$$\frac{8xy+1}{x^4y^3} dx +$$

$$\frac{1+2xy-x^3y^3}{x^3y^4} dy = 0 \quad \text{--- (2)}$$

eqⁿ (2) in form of $Mdx + Ndy = 0$

$$M = \frac{2xy+1}{x^4y^3}$$

$$N = \frac{1+2xy-x^3y^3}{x^3y^4} dy$$

General solution

$$\int M dx + \int (\text{free of } x \text{ in } N) dy = C$$

$$\int \left[\frac{2}{x^3y^2} + \frac{1}{x^4y^3} \right] dx + \int -\frac{1}{y} dy = C$$

$$\int \left[\frac{2}{y^2} \frac{x^{-3+1}}{-3+1} + \frac{1}{y^3} \frac{x^{-4+1}}{-4+1} \right] + \log y = C$$

$$\frac{2}{y^2} \frac{x^{-2}}{-2} + \frac{1}{y^3} \frac{x^{-3}}{-3} - \log y = C$$

$$-\frac{2}{y^2} \frac{1}{2x^2} + \frac{1}{3x^3y^3} - \log y = C$$

$$\frac{1}{x^2 y^2} + \frac{1}{3x^3 y^3} + \log y = -c$$

$$\frac{1}{x^2 y^2} + \frac{1}{3x^3 y^3} + \log y = c$$

7) $(x^2 y^2 + 2)y dx + (2 - 2x^2 y^2)x dy = 0$

Given that

$$(x^2 y^2 + 2)y dx + (2 - 2x^2 y^2)x dy = 0 \quad \text{--- (1)}$$

(1) is in the form of $Mdx + Ndy = 0$ where

$$M = x^2 y^3 + 2y$$

$$N = 2x - 2x^3 y^2$$

$$\frac{\partial M}{\partial y} = 3x^2 y^2 + 2$$

$$\frac{\partial N}{\partial x} = 2 - 6x^2 y^2$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

eqn (1) is not exact.

$$\text{Integrating factor} = \frac{1}{Mx - Ny}$$

$$\begin{aligned} Mx - Ny &= (x^2 y^3 + 2y)x - (2x - 2x^3 y^2)y \\ &= x^3 y^3 + 2xy - 2xy + 2x^3 y^3 \\ &= 3x^3 y^3 \end{aligned}$$

$$\frac{(x^2 y^2 + 2)y}{3x^3 y^3} dx + \frac{(2 - 2x^2 y^2)x}{3x^3 y^3} dy = 0$$

$$\frac{(x^2 y^2 + 2)}{3x^3 y^2} dx + \frac{(2 - 2x^2 y^2)}{3x^2 y^3} dy = 0$$

$$\left(\frac{1}{3x} + \frac{2}{3x^3y^2} \right) dx + \left(\frac{2}{3x^2y^2} - \frac{2}{3y} \right) dy = 0$$

(2)

② Is P_n the form of $Mdx + Ndy$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$$M_1 = \frac{1}{3x} + \frac{2}{3x^3y^2} \quad N = \frac{2}{3x^2y^2} - \frac{2}{3y}$$

then

General solution

$$\int M dx + \int (\text{free of } x \text{ in } N) dy = c$$

$$\int \left(\frac{1}{3x} + \frac{2}{3x^3y^2} \right) dx + \int -\frac{2}{3y} dy = c$$

$$\frac{1}{3} \int \frac{1}{x} dx + \frac{2}{3y^2} \int \frac{1}{x^3} dx + \frac{2}{3} \int \frac{1}{y} dy = c$$

$$\frac{1}{3} \log x + \frac{2}{3y^2} \cdot \frac{-1}{2x^2} - \frac{2}{3} \log y = c$$

$$\frac{1}{3} \log x - \frac{1}{3x^2y^2} - \frac{2}{3} \log y = c$$

$$\frac{1}{3} \left[\log x - \frac{1}{x^2y^2} - 2 \log y \right] = c$$

$$\log x - \frac{1}{x^2y^2} - \log y^2 = 3c$$

$$\log x - \log y^2 - \frac{1}{x^2y^2} = 3c$$

$$\log \left(\frac{x}{y^2} \right) - \frac{1}{x^2y^2} = k$$

$$(6). (xy \sin xy + \cos xy) y dx + (xy \sin xy - \cos xy) x dy = 0$$

Given that

$$(xy \sin xy + \cos xy) y dx + (xy \sin xy - \cos xy) x dy = 0 \quad (2)$$

① is in the form of $Mdx + Ndy = 0$.
where

$$M = (xy \sin xy + \cos xy) y \quad N = (xy \sin xy - \cos xy) x$$

$$xy^2 \sin xy + y \cos xy^2 = x^2 y \sin xy - x \cos xy$$

$$\frac{\partial M}{\partial y} = xy^2 \cos xy (x) + \sin xy (2xy) + y \sin xy (x) + \cos xy (1)$$

$$x^2 y^2 \cos xy + 2xy \sin xy + xy \sin xy + \cos xy$$

$$\frac{\partial N}{\partial x} = x^2 y \cos xy (y) + \sin xy (2xy) - x \sin xy (y) - \cos xy (1)$$

$$= x^2 y^2 \cos xy + 2xy \sin xy - xy \sin xy - \cos xy$$

$$= x^2 y^2 \cos xy - xy \sin xy - \cos xy$$

$$\left[\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \right]$$

① is not exact

$$\text{Integrating factor} = \frac{1}{Mx - Ny}$$

$$Mx - Ny = (xy^2 \sin xy + y \cos xy^2) x - (x^2 y \sin xy - x \cos xy) y$$

$$= x^2 y^2 \sin xy + xy \cos xy^2 - x^2 y \sin xy + xy \cos xy$$

$$= 2xy \cos xy$$

$$\frac{xy^2 \sin xy - y \cos xy}{2xy \cos xy} dx + \frac{x^2 y \sin xy - x \cos xy}{2xy \cos xy} dy = 0$$

$$\left(\frac{y}{2} \tan xy + \frac{1}{2x} \right) dx + \left(\frac{x}{2} \tan xy - \frac{1}{2y} \right) dy = 0$$

(2)

② P_1 is an exact D.E

Then

General Solution

$$\int M dx + \int (\text{terms free from } x \text{ in } M_1) dy = C$$

$$\int \left(\frac{y}{2} \tan xy + \frac{1}{2x} \right) dx + \int -\frac{1}{2y} dy = C$$

$$\frac{y}{2} \int \tan xy + \frac{1}{2x} \int dx + \frac{1}{2} \int \frac{1}{y} dy = C$$

$$\frac{y}{2} \frac{\log |\sec xy|}{y} + \frac{1}{2x} x - \frac{1}{2} \log x = C$$

$$\frac{\log |\sec xy|}{2} + \frac{1}{2} - \frac{1}{2} \log x = C$$

$$\frac{1}{2} [\log |\sec xy| - \log x] = C,$$

Method III

If $Mdx + Ndy = 0$ is not an exact differential equation

Case (i) :- If there exist a real value function

$$f(x) \text{ such that } \frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = f(x)$$

- then Integrating factor is $e^{\int f(x) dx}$

Case (ii) :- If there exist a real value

$$\text{function } g(y) \text{ such that } \frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = g(y)$$

- then Integrating factor is $e^{\int g(y) dy}$

① solve $(xy^2 - e^{yx^3}) dx - x^2y dy = 0$.

Given that

$$(xy^2 - e^{yx^3}) dx - x^2y dy = 0 \quad \text{--- ①}$$

① is in the form of $Mdx + Ndy = 0$

where

$$M = xy^2 - e^{yx^3}$$

$$N = -x^2y$$

$$\frac{\partial M}{\partial y} = 2xy - 0$$

$$\frac{\partial N}{\partial x} = -2xy$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

① is not an exact.

$$\text{Now consider } \frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right]$$

$$= \frac{-1}{x^2y} [-2xy + 2xy]$$

$$= \frac{-4xy}{-2xy} = \frac{-4}{x} = f(x)$$

$$\begin{aligned}
 \therefore I.F. &= e^{\int \frac{-4}{x} dx} = e^{-4 \log x} \\
 &= e^{\log x^{-4}} \\
 &= x^{-4} \\
 &= \frac{1}{x^4}
 \end{aligned}$$

Multiply (1) with $\frac{1}{x^4}$ we get

$$\frac{1}{x^4} [xy^2 - e^{yx^3}] dx - \frac{x^2 y}{x^4} dy = 0$$

$$\left[\frac{y^2}{x^3} - \frac{e^{yx^3}}{x^4} \right] dx - \frac{x^2 y}{x^4} dy = 0 \quad \text{--- (2)}$$

which is an exact D.E

General solution

$$\int M_1 dx + \int (\text{terms free from } x \text{ in } M_1) dy = C$$

$$\int \left[\frac{y^2}{x^3} + \frac{e^{yx^3}}{x^4} \right] dx + \int 0 dy = C$$

$$= y^2 \int \frac{1}{x^3} dx - \int \frac{e^{yx^3}}{x^4} dx = C$$

$$y^2 \left[\frac{-1}{2x^2} \right] + \int e^t \frac{dt}{3} = C$$

$$-\frac{y^2}{2x^2} + \frac{1}{3} e^t = C$$

$$-\frac{y^2}{2x^2} + \frac{1}{3} e^{yx^3} = C$$

$$\frac{1}{x^3} = t$$

$$\frac{-3}{x^4} dx = dt$$

$$\frac{1}{x^4} dx = -\frac{dt}{3}$$

$$2) (y^3x + y) dx + 2(xy^2 + x + y^4) dy = 0$$

Given that

$$(y^3x + y) dx + (2xy^2 + 2x + y^4) dy = 0 \quad \text{--- (1)}$$

① Is in the form of $Mdx + Ndy = 0$

$$M = y^3x + y$$

$$N = 2xy^2 + 2x + y^4$$

$$\frac{\partial M}{\partial y} = 3y^2x + 1$$

$$\frac{\partial N}{\partial x} = 4xy^2 + 2$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

① Is (not the form of) not an exact.

$$\text{Now consider } \frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = g(y)$$

$$= \frac{1}{y^3x + y} \left[4xy^2 + 2 - (3y^2x + 1) \right]$$

$$= \frac{1}{y^3x + y} \left[y^2x + 1 \right]$$

$$= \frac{y^2x + 1}{y^3x + y} \Rightarrow \frac{y^2x + 1}{y^3x + y}$$

$$= \frac{y^2x + 1}{y(y^2x + 1)} = \frac{1}{y}$$

$$\begin{aligned} \text{I.F.} &= e^{\int \frac{1}{y} dy} = e^{\log y} \\ &= e^{\log y} \\ &= y \end{aligned}$$

Multiply ① with y we get

$$y(y^3x + y)dx + (2x^2y^2 + 2x + 2y^4)y dy = 0$$

$$(y^4x + y^2)dx + (2xy^3 + 2xy + 2y^5)dy = 0 \quad \text{②}$$

where

$$M_1 = y^4x + y^2 \quad \& \quad N_1 = 2(xy^3 + xy + y^5)$$

eq ② which is an exact

$$\boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$$

General Solution

$$\int M_1 dx + \int (\text{terms free of } x \text{ in } N_1) dy = c$$

$$\int y^4x + y^2 dx + \int 2y^5 dy = c$$

$$y^4 \int x dx + y^2 \int dx + 2 \int y^5 dy = c$$

$$y^4 \left[\frac{x^2}{2} \right] + y^2(x) + \frac{y^6}{3} = c$$

$$\frac{x^2 y^4}{2} + xy^2 + \frac{y^6}{3} = c$$

③ Solve $(x^2 + y^2 + x)dx + xydy = 0$

Given that

$$(x^2 + y^2 + x)dx + xydy = 0 \quad \text{--- ①}$$

① P_5 is in the form of $Mdx + Ndy = 0$.

where

$$M = x^2 + y^2 + x$$

$$N = xy$$

$$\frac{\partial M}{\partial y} = 0 + 2y$$

$$\frac{\partial N}{\partial x} = y$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

① P_5 is not an exact

now consider $\frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = f(x)$

$$= \frac{1}{xy} [2y - y]$$

$$= \frac{y}{xy} = \frac{1}{x} = f(x)$$

$$\therefore I.F. = e^{\int f(x) dx} = e^{\int \frac{1}{x} dx}$$

$$= e^{\log x}$$

$$= x$$

Multiply ① with x we get

$$x(x^2 + y^2 + x) dx + x(xy) dy = 0$$

$$(x^3 + xy^2 + x^2) dx + x^2 y dy = 0 \quad \text{--- ②}$$

② P_5 is an exact

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

General Solution

$$\int M dx + \int (\text{term in } x^0 \text{ in } N) dy = C$$

$$\int x^4 + xy^2 + x^2 + \int 0 dy = C$$

$$\frac{x^5}{5} + y^2 \frac{x^2}{2} + \frac{x^3}{3} = C$$

$$\frac{x^5}{5} + \frac{y^2 x^2}{2} + \frac{x^3}{3} = C$$

④ Solve $y(xy + e^x) dx - e^x dy = 0$
Given that

$$y(xy + e^x) dx - e^x dy = 0 \quad \text{--- (1)}$$

(1) is in the form of $M dx + N dy = 0$
where

$$M = xy^2 + ye^x$$

$$N = -e^x$$

$$\frac{\partial M}{\partial y} = 2xy + e^x$$

$$\frac{\partial N}{\partial x} = -e^x$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

① is not an exact DE.

$$\text{now consider } = \frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = -f(x)$$

$$= \frac{1}{e^x} (xy + e^x)$$

$$= \frac{1}{e^x} (xy + e^x)$$

$$= \frac{-1(xy + e^x) + 1}{e^x}$$

Case 2)

Now consider $= \frac{1}{xy} \left[\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \right] = 0$

$$= \frac{1}{y(xy + e^x)} [-e^x - xy - e^x]$$

$$= \frac{1}{y(xy + e^x)} (-2e^x - xy)$$

$$= \frac{-2(e^x + \frac{xy}{2})}{y(xy + e^x)} = -\frac{2}{y} = g(x)$$

(multiply ① with $\frac{-2}{y}$ we get

$$\frac{-2y(xy + e^x)dx}{y} + \frac{2e^x}{y} dy = 0 \quad \text{--- ②}$$

② is an exact.

General Solution

$$\int M dx + \int (\text{term free of } x) dy = C$$

$$\int -2xy + e^x + \int 0 dy = C$$

$$-2y \int x dx + \int e^x dx = C$$

$$= -2y \frac{x^2}{2} +$$

$$I.F. = e^{\int \frac{-2}{y} dy}$$

$$= e^{\log y^2}$$

$$= \frac{1}{y^2}$$

Multiply $\frac{1}{y^2}$ with (1) we get

$$\frac{y(xy + e^x)}{y^2} dx + \frac{e^x}{y^2} dy = 0$$

$$\left[x + \frac{e^x}{y} \right] dx - \frac{e^x}{y^2} dy = 0 \quad \text{--- (2)}$$

(2) is an exact

General solution

$$\int M dx + \int (\text{terms free of } x) dy = C$$

$$\int \left(x + \frac{e^x}{y} \right) dx + \int 0 dy = C$$

$$\int x dx + \frac{1}{y} \int e^x dx + \int 0 dy = C$$

$$\frac{x^2}{2} + \frac{1}{y} e^x + C,$$

$$(5) \quad y(x+y+1)dx + x(x+3y+2)dy=0$$

Given that

$$y(x+y+1)dx + x(x+3y+2)dy=0 \quad \text{--- (1)}$$

(1) is in the form of $Mdx + Ndy=0$

where

$$M = yx + y^2 + y$$

$$N = x^2 + 3xy + 2x$$

$$\frac{\partial M}{\partial y} = x + 2y + 1$$

$$\frac{\partial N}{\partial x} = 2x + 3y + 2$$

$$\left[\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \right]$$

(1) is not an exact.

$$\text{Now consider } \frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right]$$

$$= \frac{1}{x^2 + 3xy + 2x} \left[x + 2y + 1 - 2x - 3y - 2 \right]$$

$$= \frac{1}{x^2 + 3xy + 2x} \left[-x - y - 1 \right] \neq f(x)$$

$$\text{Now consider } \frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right]$$

$$= \frac{1}{yx + y^2 + y} \left[2x + 3y + 2 - x - 2y - 1 \right]$$

$$= \frac{1}{yx + y^2 + y} [x + y + 1]$$

$$= \frac{1}{y(x + y + 1)} [x + y + 1] = \frac{1}{y}$$

$$= \frac{1}{y}$$

$$IF = e^{\int \frac{1}{y} dy}$$

$$= e^{\log y}$$

$$= y$$

Multiply y with ① we get

$$y(x + y^2 + y) y dx + (x^2 + 3xy + 2x) y dy = 0$$

$$(y^2x + y^3 + y^2) dx + (x^2y + 3xy^2 + 2xy) dy = 0 \quad \text{②}$$

② an exact D.E

$$\boxed{\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}}$$

General solution

$$\int M_1 dx + \int (\text{terms free from } x \text{ in } N_1) dy = C$$

$$\Rightarrow \int (y^3 x + y^3 + y^2) dx + \int 0 dy = C$$

$$= y^2 \int x dx + y^3 \int dx + y^2 \int dx = C$$

$$y^2 \frac{x^2}{2} + y^3 x + y^2 x = C$$

$$\frac{x^2 y^2}{2} + y^3 x + y^2 x = C.$$

$$(6). (x^3 - 2y^2) dx + 2xy dy = 0$$

Given that

$$(x^3 - 2y^2) dx + 2xy dy = 0 \quad \text{--- (1)}$$

(4) is in the form of $M dx + N dy = 0$.

where

$$M = x^3 - 2y^2$$

$$\text{or } N = 2xy$$

$$\frac{\partial M}{\partial y} = 0 - 4y$$

$$\text{or } \frac{\partial N}{\partial x} = 2y$$

$$\left[\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \right]$$

(1) is not exact

Now Consider $\frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right]$

$$= \frac{1}{2xy} \left[-4y - 2y \right]$$

$$= \frac{-6y}{2xy} = -\frac{3}{x} = f(x)$$

$$\begin{aligned}
 \mathcal{I} &= e^{\int \frac{-3}{x} dx} \\
 &= e^{-3 \int \frac{1}{x} dx} = \frac{e^{-3 \log x}}{e^{-\log x^3}} \\
 &= \frac{1}{x^3}
 \end{aligned}$$

Multiply $\frac{1}{x^3}$ with (1) we get

$$\frac{1}{x^3} (x^3 - 3y^2) dx + \frac{3xy}{x^3} dy = 0$$

$$\left[1 - \frac{3y^2}{x^3} \right] dx + \frac{3xy}{x^2} dy = 0 \quad \text{--- (2)}$$

where equation (2) is exact

Then General Solution

$$\int M dx + \int (\text{terms free of } x \text{ in } N_1) dy = C$$

$$\int \left(1 - \frac{3y^2}{x^3} \right) dx + \int 0 dy = C$$

$$\int 1 dx - \frac{3y^2}{y^2} \int \frac{1}{x^3} dx = C$$

$$x - 3y^2 \left[\frac{x^{-3+1}}{-3+1} \right] = C$$

$$x + 3y^2 \frac{x^{-2}}{3} = C$$

$$x + \frac{3y^2}{3x^2} = C$$

Method IV = Method by Inspection.

P.F. by Inspection.

In several cases the Integrating factor can be found (factor) after regrouping the terms of the equation and recognising each group as the part of an exact differential.

$$(1) \quad x dy + y dx = d(xy)$$

$$(2) \quad \frac{y dx - x dy}{y^2} = d\left(\frac{x}{y}\right)$$

$$(3) \quad \frac{x dy - y dx}{x^2} = d\left(\frac{y}{x}\right)$$

$$(4) \quad \frac{y dx - x dy}{xy} = d(\log(x/y))$$

$$(5) \quad \frac{x dy - y dx}{(xy)} = d \log(y/x)$$

$$(6) \quad \frac{y dx + x dy}{(xy)} = d(\log(xy))$$

$$(7) \quad \frac{y dx - x dy}{x^2 + y^2} = d\left(\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$(8) \quad \frac{x dy - y dx}{x^2 + y^2} = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$(9) \quad \left(y + \frac{y^3}{3} - \frac{x^2}{2}\right) dx + \frac{1}{4}(x + xy^2) dy = 0$$

$$(10) \quad (3x^2y^4 + 2xy) dx + (2x^3y^4 - x^2) dy = 0.$$

$$(7) \left(y + \frac{y^3}{3} - \frac{x^2}{2} \right) dx + \frac{1}{4} (x + xy^2) dy = 0$$

Given that

$$\left(y + \frac{y^3}{3} - \frac{x^2}{2} \right) dx + \frac{1}{4} (x + xy^2) dy = 0 \quad \text{--- (1)}$$

(1) is in the form of $Mdx + Ndy = 0$

where

$$M = y + \frac{y^3}{3} - \frac{x^2}{2} \quad N = \frac{x}{4} + \frac{xy^2}{4}$$

$$\frac{\partial M}{\partial y} = 1 + \frac{y^2}{1} = 0 \quad \frac{\partial N}{\partial x} = \frac{1}{4} + \frac{y^2}{4}$$

$$\left[\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \right]$$

where (1) is not exact

$$\text{Now consider } = \frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right]$$

$$= \frac{4}{x + xy^2} \left[1 + y^2 - \frac{1}{4} - \frac{y^2}{4} \right]$$

$$= \frac{4}{x + xy^2} \left[\frac{4 + 4y^2 - 1 - y^2}{4} \right]$$

$$= \frac{(3y^2 + 3)}{x + xy^2}$$

$$= \frac{3(y^2 + 1)}{x(1 + y^2)} = \frac{3}{x} = f(x)$$

$$\begin{aligned}
 \mathcal{I} F &= e^{\int \frac{3}{x} dx} \\
 &= e^{3 \log x} \\
 &= e^{\log x^3} \\
 &= x^3.
 \end{aligned}$$

multiply x^3 with (1) we get

$$\begin{aligned}
 x^3 \left(y + \frac{y^3}{3} - \frac{x^2}{2} \right) dx + \frac{x^3}{y} (x + xy^2) dy &= 0 \\
 \left(x^3 y + \frac{x^3 y^3}{3} - \frac{x^5}{2} \right) dx + \frac{1}{y} (x^4 + x^4 y^2) dy &= 0 \quad (2)
 \end{aligned}$$

(2) is in the form of $M dx + N dy = 0$

where $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

where (2) is an exact

then

General solution

$$\int M_1 dx + \int (\text{term free from } x \text{ in } N_1) dy = C$$

$$\int \left(x^3 y + \frac{x^3 y^3}{3} - \frac{x^5}{2} \right) dx + \int 0 dy = C$$

$$y \left(x^3 dx + \frac{y^3}{3} \int x^3 dx - \frac{1}{2} \int x^5 dx \right) = C$$

$$y \left[\frac{x^4}{4} \right] + \frac{y^3}{3} \frac{x^4}{4} - \frac{1}{2} \frac{x^6}{6} = C$$

$$\frac{x^4 y}{4} + \frac{x^4 y^3}{12} - \frac{x^6}{12} = C$$

$$3x^4 y + x^4 y^3 - x^6 = 12C$$

$$\textcircled{1} \quad x dy - y dx + 2x^3 dx = 0$$

Dividing with x^2

$$\frac{x dy - y dx}{x^2} + 2x dx = 0$$

$$\Rightarrow d\left(\frac{y}{x}\right) + 2x dx = 0$$

Taking integration

$$\int d\left(\frac{y}{x}\right) + 2 \int x dx = C$$

$$\frac{y}{x} + \cancel{x} \frac{x^2}{2} = C$$

$$\frac{y}{x} + x^2 = C,,$$

$$\textcircled{2} \quad (1+xy)y dx + (1-xy)x dy = 0$$

$$\Rightarrow y + xy^2 dx + x dy - x^2 dy = 0$$

$$y dx + x dy + xy(y dx - x dy) = 0$$

Dividing with $(xy)^2$

$$\Rightarrow \frac{y dx + x dy}{(xy)^2} + \frac{y dx - x dy}{xy} = 0$$

$$\frac{1}{(xy)^2} d(xy) + d\left[\log\left(\frac{x}{y}\right)\right] = 0$$

$$\int \frac{1}{(xy)^2} d(xy) + \int d\left[\log\left(\frac{x}{y}\right)\right] = C$$

$$\frac{-1}{xy} + \log\left(\frac{x}{y}\right) = C$$

$$\textcircled{3} \quad y(2xy + e^x) dx = e^x dy$$

$$(2xy^2 + ye^x) dx = e^x dy$$

$$2xy^2 dx + ye^x dx - e^x dy = 0$$

Dividing y^2

$$2x dx + \frac{ye^x dx - e^x dy}{y^2} = 0$$

$$\rightarrow 2x dx + d\left[\frac{e^x}{y}\right] = 0$$

$$2 \int x dx + \int d\left[\frac{e^x}{y}\right] = C$$

$$x^2 + \frac{e^x}{y} = C$$

$$x^2 + \frac{e^x}{y} = C,$$

$$\textcircled{4} \quad y(x^3 e^{xy} - y) dx + x(y + x^3 e^{xy}) dy = 0$$

$$(yx^3 e^{xy} - y^2) dx + (xy + x^4 e^{xy}) dy = 0$$

$$yx^3 e^{xy} dx - y^2 dx + xy dy + x^4 e^{xy} dy = 0$$

$$x^3 e^{xy} [y dx + x dy] + y [x dy - y dx] = 0$$

$$e^{xy} [y dx + x dy] + \frac{y}{x} \frac{x dy - y dx}{x^2} = 0$$

$$e^{xy} d(xy) + \frac{y}{x} d\left(\frac{y}{x}\right) = 0$$

$$\int e^{xy} d(xy) + \int \frac{y}{x} d\left(\frac{y}{x}\right) = C$$

$$e^{xy} + \frac{1}{2} \left(\frac{y}{x}\right)^2 = C$$

$$(5) \quad xdy - ydx + a(x^2 + y^2)dx = 0$$

$$\frac{xdy - ydx}{x^2 + y^2} + a dx = 0$$

$$d\left(-\tan^{-1} \frac{y}{x}\right) + a dx = 0$$

$$\int d\left(-\tan^{-1} \frac{y}{x}\right) + \int a dx = C$$

$$-\tan^{-1}\left(\frac{y}{x}\right) + ax = C$$

$$(6) \quad y dx - x dy + 3x^2 y^2 e^{x^3} dx = 0$$

$$\frac{y dx - x dy}{y^2} + 3x^2 e^{x^3} dx = 0$$

$$d\left(\frac{x}{y}\right) + 3x^2 e^{x^3} dx = 0$$

Integration.

$$\int d\left(\frac{x}{y}\right) + \int 3x^2 e^{x^3} dx = C$$

$$\left(\frac{x}{y} + \frac{x^3}{3} e^{x^3}\right) = C$$

$$\frac{x}{y} + x^3 e^{x^3} = C$$

$$\frac{x}{y} + \int e^t dt = C$$

$$\frac{x}{y} + e^t = C$$

$$\frac{x}{y} + e^{x^3} = C$$

$$(7) \quad xdy - ydx = xy^2 dx$$

$$xdy - ydx - xy^2 dx = 0$$

$$xdy - ydx - xy^2 dx = 0$$

$$-(ydx - xdy) + xy^2 dx = 0$$

$$\frac{ydx - xdy}{y^2} + x dx = 0$$

$$\int d\left(\frac{x}{y}\right) + x dx = 0$$

$$\int d\left(\frac{x}{y}\right) + \int x dx = C$$

$$\frac{x}{y} + \frac{x^2}{2} = C$$

$$(8) \quad xdy - (y-x)dx = 0.$$

$$xdy - (y-x)dx = 0.$$

$$xdy - ydx + xdx = 0.$$

$$\frac{xdy - ydx}{x^2} + \frac{1}{x} dx = 0.$$

$$d\left(\frac{y}{x}\right) + \frac{1}{x} dx = 0$$

$$\int d\left(\frac{y}{x}\right) + \int \frac{1}{x} dx = C$$

$$\frac{y}{x} + \log x = C.$$

4
x=dt

Linear Differential Equation

In a DE if the dependent variable and its Differential Coefficient appears (occurs) P_n

the first degree only and not multiplied together then the differential equation is called linear differential equation.

Standard LDE in y is

$$\frac{dy}{dx} + p(x)y = Q(x) \quad \text{LDE in y}$$

then I.F is

$$y(I.F) = \int (I.F) Q(x) dx + C$$

where $I.F = e^{\int p(x) dx}$

standard LDE in x is

$$\frac{dx}{dy} + p(y)x = Q(y) \quad \text{LDE in x}$$

$$x(I.F) = \int (I.F) Q(y) dy + C$$

where $I.F = e^{\int p(y) dy}$

① Solve $(x+1) \frac{dy}{dx} - y = e^{3x}(x+1)^2$

Given that

$$(x+1) \frac{dy}{dx} - y = e^{3x}(x+1)^2 \quad \text{--- ①}$$

Divide $(x+1)$

we get

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1) \quad \text{--- ②}$$

P_n LDE in y

where $p(x) = \frac{-1}{x+1}$ $Q(x) = e^{3x}(x+1)$

$$\begin{aligned}
 I.F. &= e^{\int \frac{1}{x+1} dx} \\
 &= e^{-\log(x+1)} \\
 &= e^{\log(x+1)^{-1}} \\
 &= \frac{1}{x+1}
 \end{aligned}$$

then G.S.

$$y(I.F.) = \int (I.F.) Q(x) dx + C$$

$$y\left(\frac{1}{x+1}\right) = \int \frac{1}{x+1} e^{3x} dx + C$$

$$\frac{y}{x+1} = \log(x+1) \int e^{3x} dx + C$$

$$\frac{y}{x+1} = \frac{e^{3x}}{3} + C$$

$$y = \frac{x+1}{3} e^{3x} + C(x+1),$$

$$y \log y dx + (x - \log y) dy = 0.$$

Given that

$$y \log y dx + (x - \log y) dy = 0 \quad \text{--- (1)}$$

$$y \log y dx = -x + \log y dy$$

$$y \log y \frac{dx}{dy} + x = \log y$$

Divided by with $y \log y$

$$\frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y} \quad \text{--- (2)}$$

Ps. $\log P_n x$

where

$$P(y) = \frac{1}{y \log y}$$

$$Q(x) = \frac{1}{y}$$

$$\begin{aligned}
 I.F &= e^{\int p(y) dy} \quad \text{put } \log y = z \\
 &= e^{\int \frac{1}{y \log y} dy} \quad \frac{1}{y} dz = dz \\
 &= e^{\int \frac{1}{z} dz} = e^{\log z} = z = \log y
 \end{aligned}$$

$$x(I.F) = \int (I.F) Q(x) dx + C$$

$$x(\log y) = \int \log y \cdot \frac{1}{y} dx + C$$

$$x \log y = \int z dz + C$$

$$x \log y = \frac{z^2}{2} + C$$

$$x \log y = \frac{(\log y)^2}{2} + C$$

$$x = \frac{1}{2} \log y + \frac{C}{\log y}$$

$$(5) (1-x^2) \frac{dy}{dx} - xy = 1$$

Given that

$$(1-x^2) \frac{dy}{dx} - xy = 1$$

Divided with $(1-x^2)$

$$\frac{dy}{dx} - \frac{xy}{(1-x^2)} = \frac{1}{1-x^2}$$

LDE in y

where

$$p(x) = \frac{-x}{(1-x^2)}$$

$$Q(x) = \frac{1}{1-x^2}$$

$$\begin{aligned}
 \mathcal{I}F &= e^{\int \frac{x}{(1-x^2)} dx} \\
 &= e^{\int -x \cdot \frac{1}{(1-x^2)} dx} \\
 &= e^{\int \frac{-2x}{2(1-x^2)} dx} \\
 &= e^{\int \frac{-2x}{(1-x^2)} dx} \\
 &= e^{\frac{1}{2} \log(1-x^2) dx} \\
 &= e^{\log(1-x^2)^{1/2}} \\
 &= (1-x^2)^{1/2}
 \end{aligned}$$

Q. 6.

$$y(\mathcal{I}F) = \int \mathcal{I}F Q(x) dx + C$$

$$y(1-x^2)^{1/2} = \int (1-x^2)^{1/2} \frac{1}{(1-x^2)} dx + C$$

$$y(1-x^2)^{1/2} = \int \frac{1}{(1-x^2)^{1/2}} dx + C$$

$$= \int \frac{1}{\sqrt{1-x^2}} dx + C$$

$$y(1-x^2)^{1/2} = \sin^{-1} x + C$$

$$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} + \frac{C}{\sqrt{1-x^2}}$$

(4) $\cos^2 x \frac{dy}{dx} + y = -\tan x$

Divide with $\cos^2 x$

$$\frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{-\tan x}{\cos^2 x} \quad \text{--- (2)}$$

P_n LDE $P_n y$

where

$$p(x) = \frac{1}{\cos^2 x} \\ = \sec^2 x$$

$$Q(x) = \frac{-\tan x}{\cos^2 x} \\ = -\tan x \sec^2 x$$

$$\begin{aligned} I.P. &= e^{\int p(x) dx} \\ &= e^{\int \frac{1}{\cos^2 x} dx} \\ &= e^{\int \sec^2 x dx} \\ &= e^{\tan x} \end{aligned}$$

$$G.I.S. =$$

$$y(I.P.) = \int I.P. Q(x) dx + C$$

$$y(e^{\tan x}) = \int e^{\tan x} (-\tan x \sec^2 x) dx + C$$

$$ye^{\tan x} = \int e^z \cdot z dz + C$$

$$\begin{aligned} \tan x &= z \\ \sec^2 x dx &= dz \end{aligned}$$

$$ye^{\tan x} = e^z (z-1) + C$$

$$\Rightarrow ye^{\tan x} = e^{\tan x} (\tan x - 1) + C$$

$$\Rightarrow y = (\tan x - 1) + \frac{C}{e^{\tan x}} //$$

$$1) dr + (2r \cot \theta + \sin 2\theta) d\theta = 0$$

$$dr + (2r \cot \theta + \sin 2\theta) d\theta = 0 \quad \text{--- (1)}$$

$$dr = -(2r \cot \theta + \sin 2\theta) d\theta$$

$$\frac{dr}{d\theta} + 2r \cot \theta = -\sin 2\theta \quad \text{--- (2)}$$

$$P_n \text{ and } P_n r$$

$$\frac{dr}{d\theta} + p(\theta)r = Q(\theta)$$

where

$$p(\theta) = 2 \cot \theta, \quad Q(\theta) = -\sin 2\theta$$

$$I.F. = e^{\int p(\theta) d\theta}$$

$$= e^{\int 2 \cot \theta \cdot d\theta}$$

$$= e^{2 \log \sin \theta}$$

$$= e^{\log (\sin \theta)^2}$$

$$= \sin^2 \theta$$

G.S

$$r(I.F.) = \int I.F. Q(\theta) d\theta + C$$

$$r \sin^2 \theta = \int \sin^2 \theta (-\sin 2\theta) d\theta + C$$

$$r \sin^2 \theta = -\int \sin^2 \theta (2 \sin \theta \cos \theta) d\theta + C$$

$$r \sin^2 \theta = -2 \int (\sin^3 \theta \cos \theta) d\theta + C$$

$$r \sin^2 \theta = \frac{-(\sin^4 \theta)}{\frac{4}{4} \cdot 2} + C$$

$$r = \frac{\sin^2 \theta}{2} + \frac{C}{\sin^2 \theta}$$

$$r = \frac{1}{2} \sin^2 \theta + C \operatorname{cosec}^2 \theta,$$

$$\int f'(\theta) [f(\theta)]^n d\theta = \frac{f(\theta)^{n+1}}{n+1}$$

$$\frac{dy}{dx} = - \left(\frac{x - y \cos x}{1 + \sin x} \right)$$

$$\frac{dy}{dx} + \frac{y \cos x}{1 + \sin x} = \frac{-x}{1 + \sin x}$$

It is in the form of $\frac{dy}{dx} + p(x)y = q(x)$

$$\frac{dy}{dx} + p(x)y = q(x)$$

$$p(x) = \frac{\cos x}{1 + \sin x}, \quad q(x) = \frac{-x}{1 + \sin x}$$

where

$$I.F = e^{\int p(x) dx}$$

$$= e^{\int \frac{\cos x}{1 + \sin x} dx}$$

$$= e^{\log(1 + \sin x)}$$

$$= 1 + \sin x$$

Gr 8

$$y(1 + \sin x) = \int (1 + \sin x) \frac{-x}{1 + \sin x} dx + C$$

$$y(1 + \sin x) = -\int x + C$$

$$y(1 + \sin x) = -\frac{x^2}{2} + C$$

$$y = \frac{-x^2}{2(1 + \sin x)} + \frac{C}{1 + \sin x}$$

$$\textcircled{2} \quad \frac{dy}{dx} + xy = 2x^2$$

$$\frac{dy}{dx} + xy = 2x^2 e^{-x^2}$$

form of LDE in y

$$\frac{dy}{dx} + p(x)y = Q(x)$$

$$p(x) = x \quad Q(x) = 2x^2 e^{-x^2}$$

where

$$I.F = e^{\int p(x) dx}$$

$$= e^{\int x dx}$$

$$= e^{\frac{x^2}{2}}$$

$$= e^{x^2}$$

G.S

$$y(I.F) = \int I.F \cdot Q(x) dx + C$$

$$y(e^{x^2}) = \int e^{x^2} \cdot 2x^2 e^{-x^2} dx + C$$

$$= 2 \int e^{x^2} e^{-x^2} dx + C$$

$$= 2 \int \frac{e^{x^2}}{e^{x^2}} dx + C = 2 \int 1 dx + C$$

$$y(e^{x^2}) = 2x + C$$

$$y = \frac{2x}{e^{x^2}} + \frac{C}{e^{x^2}}$$

$$(2) (1+y^2)dx + (x - e^{-\tan^{-1}y})dy = 0$$

$$(1+y^2)dx = -(x - e^{-\tan^{-1}y})dy$$

$$(1+y^2) \frac{dx}{dy} = -x + e^{-\tan^{-1}y}$$

$$(1+y^2) \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{-\tan^{-1}y}}{1+y^2}$$

Divide with $(1+y^2)$

$$\frac{dx}{dy} + \frac{x}{(1+y^2)} = \frac{e^{-\tan^{-1}y}}{1+y^2}$$

is in the form of

$$\frac{dx}{dy} + p(y)x = Q(y)$$

$$p(y) = \frac{1}{1+y^2}, \quad Q(y) = \frac{e^{-\tan^{-1}y}}{1+y^2}$$

$$I.F. = e^{\int p(y) dy}$$

$$= e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1}y}$$

$$= \frac{e^{\log(1+y^2)}}{(1+y^2)}$$

Gr. S.

$$x(1+y^2) = \int I.F. Q(y) dy + C$$

$$x(1+y^2) = \int (1+y^2) e^{-\tan^{-1}y} dy + C$$

Ex. 5.

$$x(1+y^2)$$

$$x e^{-\tan^{-1} y} = \int \tan^{-1} y \frac{e^{-\tan^{-1} y}}{1+y^2} dy + C$$

$$x e^{-\tan^{-1} y} = \int \frac{e^{-\tan^{-1} y}}{1+y^2} dy + C$$

$$x e^{-\tan^{-1} y} = \int \frac{1}{1+y^2} dy + C$$

$$x e^{-\tan^{-1} y} = -\tan^{-1} y + C$$

$$x e^{-\tan^{-1} y} = -\tan^{-1} y + C,$$

⇒ Equations reducible to LDE :-

Method 1 :- A D.E of the form $f'(y) \frac{dy}{dx} + f(y)p = q$ ①

where p & q are either functions of x or constant

To reduce equation ① to linear

take

$$f(y) = z$$

then

$$f'(y) \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} + pz = q \text{ --- ② } p \text{ is a D.E in } z$$

$$\text{Ex. 5 } z \text{ (I.F)} = \int \text{(I.F)} q(x) dx + C$$

And G.I. of given eqⁿ obtain by substituting

$$z = f(y)$$

Method 2: Bernoulli's equation.

An equation of the form $\frac{dy}{dx} + py = Qy^n$ is called the Bernoulli's equation where

P & Q are either functions of x or constant

Case (i), $n=1$ $\Rightarrow \frac{dy}{dx} + py = Qy$

$$\frac{dy}{dx} + (p-Q)y = 0$$

$$\frac{dy}{dx} + (p-Q)dx = 0$$

Case (ii), To reduce eqn (i) to linear divide

It with y^n

we get

$$y^{-n} \frac{dy}{dx} + py^{1-n} = Q \quad (2)$$

Take $y^{1-n} = z$

$$(1-n)y^{1-n} \frac{dy}{dx} = \frac{dz}{dx}$$

$$(1-n)y^{-n} \frac{dy}{dx} = \frac{dz}{dx}$$

$$y^{-n} \frac{dy}{dx} = \frac{1}{1-n} \frac{dz}{dx}$$

$$\frac{1}{1-n} \frac{dz}{dx} + pz = Q$$

$$\frac{dz}{dx} + (1-n)pz = Q \quad \text{is a LDE in } z$$

It is

$$z(I.P) = \int (I.P) Q(x) dx + C.$$

$$) \frac{dy}{dx} = \frac{-\tan y}{1+x} = (1+x)e^x \sec y.$$

Given that

$$\frac{dy}{dx} = \frac{-\tan y}{1+x} = (1+x)e^x \sec y \quad \text{--- (1)}$$

① is not a linear equation

And can be reducible to linear

Divide eqⁿ ① with $\sec y$

$$\frac{1}{\sec y} \frac{dy}{dx} = \frac{-\tan y}{(1+x) \sec y} = (1+x)e^x$$

$$\cos y \frac{dy}{dx} = \cancel{\cos y} \frac{\sin y}{\cancel{\cos y} (1+x)} = (1+x)e^x$$

$$\cos y \frac{dy}{dx} = \frac{\sin y}{1+x} = (1+x)e^x \quad \text{--- (2)}$$

is of the form $f(y) \frac{dy}{dx} + f(y)p(x) = Q(x)$

$$f(y) = \sin y$$

$$\sin y = z$$

$$\cos y \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} = \frac{z}{1+x} = (1+x)e^x \quad \text{--- (3) is a DE in } z$$

P.e ③ is in the form $\frac{dz}{dx} + p(x)z = Q(x)$

$$p(x) = \frac{-1}{1+x}, \quad Q(x) = (1+x)e^x$$

$$\therefore I.F = e^{\int p(x) dx}$$

$$= e^{\int \frac{-1}{1+x} dx}$$

$$= e^{-\log(1+x)}$$

$$= e^{\log(1+x)^{-1}}$$

$$= \frac{1}{1+x}$$

General solution of equation (3) is

$$x(I.F) = \int (I.F) Q(x) dx + C$$

$$x \cdot \frac{1}{1+x} = \int \frac{1}{1+x} (1+x) e^x dx + C$$

$$\frac{x}{1+x} = \int e^x dx + C$$

$$\frac{x}{1+x} = e^x + C$$

put $x = \sin y$

G.S of (1) is

$$\frac{\sin y}{1+\sin y} = e^x + C$$

$$\sin y = e^x(1+\sin y) + C(1+\sin y)$$

$$(2) \left(e^y \left(\frac{dy}{dx} + 1 \right) \right) = e^x$$

Given that $\left(e^y \left(\frac{dy}{dx} + 1 \right) \right) = e^x$

$$e^y \frac{dy}{dx} + e^y = e^x \quad \text{--- (1)}$$

① is not linear equation

— And can be reducible to linear

if it is in Bernoulli's equation

$$e^y \frac{dy}{dx} + e^y = e^x$$

not

② $e^y \left(\frac{dy}{dx} + 1 \right) = e^x$

$$e^y \frac{dy}{dx} + e^y = e^x \quad \text{--- ①}$$

① is of the form of $f(y) \frac{dy}{dx} + g(y) = Q(x)$

$$\text{let } e^y = z$$

$$e^y \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} + z = e^x \quad \text{--- ②} \quad \text{is a DE in } z$$

i.e. ② is in the form $\frac{dz}{dx} + p(x)z = Q(x)$

$$p(x) = 1$$

$$Q(x) = e^x$$

$$\begin{aligned} \therefore \text{IF} &= e^{\int dx} \\ &= e^{\frac{x^2}{2}} \\ &= e^x \end{aligned}$$

Ge. s. ② is

$$z(\text{I.F.}) = \int (\text{I.F.}) Q(x) dx + C$$

$$z(e^x) = \int e^x e^x dx + C$$

$$z(e^x) = \int e^{2x} dx + C$$

$$ze^x = \frac{e^{2x}}{2} + C$$

$$e^y e^x = \frac{e^{2x}}{2} + C$$

$$e^y = \frac{1}{2} e^x + C e^x$$

$$(3) \quad x \frac{dy}{dx} + y = x^3 y^6$$

Given that

$$x \frac{dy}{dx} + y = x^3 y^6$$

Divide with y^6

$$\frac{x}{y^6} \frac{dy}{dx} + y^{-5} = x^3$$

$$x y^{-6} \frac{dy}{dx} + y^{-5} = x^3$$

$$(y^{-5} = z$$

$$-5 y^{-6} \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{1}{y^6} \frac{dy}{dx} + y^{-5} x^1 = x^2$$

$$y^{-6} \frac{dy}{dx} + \frac{y^5}{x} = x^2 \quad (2)$$

is in the form of $y^{-n} \frac{dy}{dx} + p y^{-n} = Q$

$$y^{-5} = z$$

$$-5 y^{-6} \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow y^6 \frac{dy}{dx} = -\frac{1}{5} \frac{dz}{dx}$$

$$-\frac{1}{5} \frac{dz}{dx} + \frac{z^4}{x} = x^2 \quad ?$$

$$\frac{dz}{dx} - \frac{5z}{x} = -5x^2$$

Q1 S.

$$I.F = e^{\int p(x) dx}$$

$$= e^{-5 \int \frac{1}{x}}$$

$$= e^{-5 \int \frac{1}{x}}$$

$$= e^{\log x^{-5}}$$

$$= x^{-5}$$

Q1 S

$$x(I.F) = \int (I.F) Q(x) dx + C$$

$$x x^{-5} = \int x^{-5} - 5x^2 dx + C.$$

$$x x^{-5} = -5 \int x^{-3} dx + C$$

$$x x^{-5} = -5 \frac{x^{-2}}{-2} + C$$

$$x x^{-5} = \frac{5 x^{-2}}{2} + C$$

put $x = y^{-5}$

$$y^{-5} x^{-5} = \frac{5 x^{-2}}{2} + C.$$

$$y^{-6} = \frac{5}{2} x^3 + C x^{-5}$$

$$(7) \quad \frac{dy}{dx} + \frac{y}{x} = y^3 x \quad / \quad \frac{dy}{dx} + \frac{y}{x} = y^3$$

Div

$$\frac{dy}{dx} + \frac{y}{x} = y^3$$

Divide with y^2

$$\frac{1}{y^3} \frac{dy}{dx} + \frac{1}{xy^2} = x$$

$$y^{-3} \frac{dy}{dx} + x^{-1} y^{-2} = x \quad \text{--- (2)}$$

is in the form of $y^{-n} \frac{dy}{dx} + P y^{-n} = Q$

$$\text{Take } y^{-2} = z$$

$$-2 y^{-3} \frac{dy}{dx} = \frac{dz}{dx}$$

$$y^{-3} \frac{dy}{dx} = \frac{-1}{2} \frac{dz}{dx}$$

$$-\frac{1}{2} \frac{dz}{dx} + x^{-1} z = x$$

$$\frac{dz}{dx} - 2x^{-1} z = x$$

$$P(x) = -\frac{2}{x} \quad Q = x$$

$$\begin{aligned} IF &= \int e^{\int P(x) dx} \\ &= e^{\int -\frac{2}{x} dx} \end{aligned}$$

$$= \int \frac{1}{x} dx$$

$$= \log x$$

$$= \log x^2$$

$$= x^{-2}$$

Q. 6.

$$x(I.F) = \int I.F. Q(x) dx + C$$

$$x(x^{-2}) = \int x^{-2} x dx + C$$

$$x(x^{-2}) = \int x^{-1} dx + C$$

$$x x^{-2} = \int \frac{1}{x} dx + C$$

$$x x^{-2} = \log x + C$$

put $x = y^{-2}$

$$y^{-2} x^{-2} = \log x + C$$

$$\frac{1}{y^2}$$

Q. 7. $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Given that

$$\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y \quad \text{--- (1)}$$

Q. 8. is not linear can be reducible to linear.

divide with $\cos^2 y$

$$\frac{1}{\cos^2 y} \frac{dy}{dx} + x \frac{\sin 2y}{\cos^2 y} = x^3$$

$$\sec^2 y \frac{dy}{dx} + x \frac{2 \sin y \cos y}{\cos^2 y} = x^3$$

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \quad \text{--- (2)}$$

(2) is of the form $f'(y) \frac{dy}{dx} + f(y) = Q(x)$

$$\tan y = z$$

$$\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} + 2x z = x^3 \quad \text{--- (3)}$$

(3) is linear in z

$$P(x) = 2x \quad Q(x) = x^3$$

$$\text{I.F.} = e^{\int P(x) dx}$$

$$= e^{\int 2x dx}$$

$$= e^{x^2}$$

$$= e^{x^2}$$

$$= e^{x^2}$$

$$= e^{x^2}$$

G.S.

$$z(\text{I.F.}) = \int \text{I.F.} Q(x) dx + C.$$

$$z e^{x^2} = \int e^{x^2} x^3 dx + C$$

$$z e^{x^2} = \int e^{x^2} x^2 \cdot x dx + C.$$

$$z e^{x^2} = \int e^{x^2} x \frac{dt}{2} + C$$

$$[x^2 = t]$$

$$2x dx = dt$$

$$x dx = \frac{dt}{2}$$

$$ze^{x^2} = \frac{1}{2} \int e^z z \, dz + C$$

$$ze^{x^2} = \frac{1}{2} \int e^z (z-1) \, dz + C$$

$$ze^{x^2} = \frac{e^{x^2}}{2} (x^2 - 1) + C$$

$$\text{put } z = -\tanh$$

$$-\tanh e^{x^2} = \frac{e^{x^2}}{2} (x^2 - 1) + C$$

$$-\tanh = \frac{1}{2} (x^2 - 1) + C,$$

$$\textcircled{6} \quad \frac{dy}{dx} + y \cot x = y^2 \sin^2 x \cos^2 x$$

Given that

$$\frac{dy}{dx} + y \cot x = y^2 \sin^2 x \cos^2 x \quad \text{--- (1)}$$

Mulde with y^2

$$\frac{1}{y^2} \frac{dy}{dx} + \frac{\cot x}{y} = \sin^2 x \cos^2 x$$

$$y^{-2} \frac{dy}{dx} + y^{-1} \cot x = \sin^2 x \cos^2 x \quad \text{--- (2)}$$

is in the form of $y^{-n} \frac{dy}{dx} + p y^{-n} = q$

$$y^{-1} = z$$

$$-y^{-2} \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dy}{dz} = \frac{1}{y^2} \, dz$$

$$y^{-2} \frac{dy}{dz} = - \frac{dz}{dz}$$

$$- \frac{dz}{dx} + \frac{z^2}{x} = -\frac{\sin^2 x \cos^2 x}{x}$$

$$\frac{dz}{dx} - \frac{z^2}{x} = -\frac{\sin^2 x \cos^2 x}{x}$$

$$p(x) = \frac{z^2}{x} \quad Q(x) = -\frac{\sin^2 x \cos^2 x}{x}$$

$$\begin{aligned} I.F. &= e^{\int p(x) dx} \\ &= e^{\int \frac{z^2}{x} dx} \\ &= e^{-\log(\sin x)} \\ &= \frac{1}{\sin x} = \operatorname{cosec} x \end{aligned}$$

Q. 4

$$I.F. = \int I.F. Q(x) dx + C$$

$$\frac{x}{\sin x} = \int \frac{1}{\sin x} (\sin^2 x \cos^2 x) dx + C$$

$$\frac{x}{\sin x} = \int (\sin x) \cos^2 x dx + C$$

$$\frac{x}{\sin x} = \frac{(\cos x)^3}{3} + C$$

$$\int f'(x) [f(x)]^n dx$$

$$= \frac{[f(x)]^{n+1}}{n+1}$$

put $x = y^{-1}$

$$\frac{y^1}{\sin x} = \frac{(\cos x)^3}{3} + C$$

$$y^1 = \sin x \left(\frac{(\cos x)^3}{3} + \sin x C \right)$$

$$7) \frac{dy}{dx} + \frac{y}{x-1} = xy^{1/3}$$

Given that

$$\frac{dy}{dx} + \frac{y}{x-1} = xy^{1/3} \quad \text{--- (1)}$$

divide (1) with $y^{1/3}$

$$y^{1/3} \frac{dy}{dx} + \frac{y^{2/3}}{x-1} = x$$

Take $y^{1/3} = z$

$$\frac{2}{3} y^{-1/3} \frac{dy}{dx} = \frac{dz}{dx}$$

$$y^{-1/3} \frac{dy}{dx} = \frac{3}{2} \frac{dz}{dx}$$

$$\frac{3}{2} \frac{dz}{dx} + \frac{z}{x-1} = x$$

$$\frac{dz}{dx} + \frac{2}{3} \frac{z}{x-1} = \frac{2x}{3} \quad \text{--- (2)}$$

where

$$P(x) = \frac{2}{3(x-1)}$$

$$Q = \frac{2x}{3}$$

$$I.F. = e^{\int \frac{2}{3(x-1)} dx}$$

$$= e^{\frac{2}{3} \int \frac{1}{x-1} dx}$$

$$= e^{\frac{2}{3} \log(x-1)}$$

$$= e^{\log(x-1)^{2/3}}$$

$$= (x-1)^{2/3}$$

Q. c.

$$x(TF) = \int TF Q(x) dx + C$$

$$x(x-1)^{2/3} = \int \frac{2}{3} x(x-1)^{2/3} dx + C$$

$$x(x-1)^{2/3} = \frac{2}{3} \int x(x-1)^{2/3} dx + C$$

$$= \frac{2}{3} \left[x \frac{(x-1)^{5/3}}{5/3} - \frac{(x-1)^{5/3}}{8/3} \right] + C$$

$$x(x-1)^{2/3} = \frac{2}{3} \times \frac{3}{5} x(x-1)^{5/3} - \frac{2}{3} \times \frac{3}{8} (x-1)^{5/3}$$

$$x = \frac{2}{5} x(x-1) - \frac{1}{4} (x-1)^2 + \frac{C}{(x-1)^{2/3}}$$

$$\text{Put } x = y^{2/3}$$

$$\therefore y^{2/3} = \frac{2}{5} x(x-1) - \frac{1}{4} (x-1)^2 + \frac{C}{(x-1)^{2/3}}$$

$$\textcircled{8} \quad \cos x \, dy = y(\sin x - y) \, dx$$

$$\cos x \, dy = (y \sin x - y^2) \, dx$$

$$\cos x \frac{dy}{dx} = y \sin x - y^2$$

$$\cos x \frac{dy}{dx} - y \sin x = -y^2$$

Divide with $\cos x$ and $-y^2$

$$\frac{1}{-y^2} \frac{dy}{dx} + \frac{\sin x}{y \cos x} = \frac{1}{\cos x}$$

$$-y^{-2} \frac{dy}{dx} + y^{-1} \frac{\sin x}{\cos x} = \frac{1}{\cos x} \quad (2)$$

(2) is in the form of $y^{-n} \frac{dy}{dx} + py^{-n} = Q$

$$y^{-2} \frac{dy}{dx} + y^{-1} \frac{\sin x}{\cos x} = \frac{1}{\cos x}$$

$$y^{-2} \frac{dy}{dx} + y^{-1} \tan x = \frac{1}{\cos x}$$

$$y^{-2} \frac{dy}{dx} + y^{-1} \tan x = -\sec x$$

$$p(x) = -\tan x$$

$$Q(x) = -\sec x$$

$$I.F. = e^{\int \tan x dx}$$

$$e^{\log(\sec x)}$$

Take $y^{-1} = z$

$$-1 y^{-2} \frac{dy}{dx} = \frac{dz}{dx}$$

$$-\frac{dz}{dx} + y^{-1} \tan x = -\sec x$$

$$\frac{dz}{dx} - y^{-1} \tan x = \sec x$$

$$p(x) = -\tan x$$

$$Q(x) = \sec x$$

$$I.F. = e^{\int p(x) dx}$$

$$= e^{\int -\tan x dx}$$

$$= e^{-\log(\sec x)}$$

$$= e^{-1} = \frac{1}{\sec x}$$

$$= \cos x$$

Q 5.

$$I(1P) = \int IF Q(x) dx + C$$

$$x \cos x = \int \cos x (-\sec x) dx + C.$$

$$x \cos x = - \int \cos x \sec x dx + C$$

$$x \cos x = - \int \frac{1}{\sec x} \sec x dx + C$$

$$x \cos x = - \int dx + C$$

$$x \cos x = -x + C$$

$$x = \frac{1}{y}$$

$$\frac{1}{y} \cos x = -x + C,$$

$$(9) \quad \frac{dy}{dx} = (\sin x - \sin y) \frac{\cos x}{\cos y}$$

$$(10) \quad 2 \frac{dy}{dx} - y \sec x = y^3 \tan x$$

$$(11) \quad \cos^2 x \frac{dy}{dx} + y = -\tan x$$

$$(12) \quad dr + (2r \cos \theta + \sin 2\theta) d\theta = 0$$

Newton's Law of Cooling

Statement: The rate of change of temperature of a body is directly proportional to the difference between temperature of body and its surrounding media. That is θ is the temperature of the body and θ_0 is the surrounding medium. Then the differential equation is given by

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -k(\theta - \theta_0) \quad \text{(where } -k \text{ is the proportional constant)}$$

$$\Rightarrow \frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\Rightarrow \frac{d\theta}{\theta - \theta_0} = -k dt \quad \text{Taking integration}$$

$$\int \frac{1}{\theta - \theta_0} d\theta = -k \int dt + C$$

$$\log(\theta - \theta_0) = -kt + C \quad \text{--- (1)}$$

Initially ($t=0$), $\theta = \theta_1$

$$\Rightarrow \log(\theta_1 - \theta_0) = C$$

Sub C in (1)

$$\log(\theta - \theta_0) = -kt + \log(\theta_1 - \theta_0)$$

$$\log \left[\frac{\theta - \theta_0}{\theta_1 - \theta_0} \right] = -kt$$

$$\frac{\theta - \theta_0}{\theta_1 - \theta_0} = e^{-kt}$$

$$\theta - \theta_0 = (\theta_1 - \theta_0) e^{-kt}$$

$$\boxed{\theta = \theta_0 + (\theta_1 - \theta_0) e^{-kt}} \quad \theta_1 \text{ is the}$$

required temperature of the body at any time t or given time (t)

① The temperature of ^{the} surrounding air is 20°C and temperature of a hot body reduces from 100 to 70°C in 1 hour. Find the temperature of the body after 2 hours.

Given that temperature of the air $\theta_0 = 20^\circ\text{C}$

$$\text{when } t = 0, \theta = 100^\circ\text{C}$$

$$t = 1 \text{ hr}, \theta = 70^\circ\text{C}$$

$$t = 2 \text{ hr}, \theta = ?$$

According to Newton's law of cooling, we have that the rate of change of temperature of a body is proportional to the difference between temperature of body and its surrounding medium.

$$\Rightarrow \frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\Rightarrow \frac{d\theta}{\theta - 20} = -k dt$$

$$\Rightarrow \int \frac{1}{\theta - 20} d\theta = -k \int dt + C$$

$$\Rightarrow \log(\theta - 20) = -kt + C \quad \text{--- (1)}$$

G.T. when $t = 0, \theta = 100$

$$\text{(1)} \Rightarrow \log(100 - 20) = C$$

$$\log(80) = C$$

Substitute C in eqⁿ (1).

$$\log(\theta - 20) = -kt + \log 80$$

$$\log \left[\frac{\theta - 20}{80} \right] = -kt \quad \text{--- (2)}$$

Also G-T $t = 3 \text{ hr}$ $\theta = 40^\circ\text{C}$

$$\textcircled{2} \Rightarrow \log \left[\frac{\theta - 20}{80} \right] = -k(t) \quad (1)$$

$$\log \frac{50}{80} = -k$$

$$\log \frac{5}{8} = -k$$

Substitute k in $\textcircled{1}$.

$$\log \left[\frac{\theta - 20}{80} \right] = \log \left(\frac{5}{8} \right) \times t \quad \text{---} \textcircled{3}$$

when $-t = 2 \text{ hr}$

$$\textcircled{3} \Rightarrow \log \left[\frac{\theta - 20}{80} \right] = \log \left[\frac{5}{8} \right] \times 2$$

$$\log \left[\frac{\theta - 20}{80} \right] = 2 \log \left[\frac{5}{8} \right]$$

$$\log \left[\frac{\theta - 20}{80} \right] = \log \left(\frac{5}{8} \right)^2$$

$$\theta - 20 = 80 \left(\frac{5}{8} \right)^2$$

$$\theta - 20 = 80 \left(\frac{25}{64} \right)$$

$$\theta - 20 = \frac{3125}{8}$$

$$\theta - 20 = 31.25$$

$$\theta = 31.25 + 20$$

$$\theta = 51.25^\circ\text{C}$$

After two hours temperature of body is 51.25°C

② If the air is maintained at 30°C and temperature of body cools from 80 to 60°C in 12 min. find the temperature of body after 24 mins.

③ The temperature of the hot body reduces from 120°C to 80°C in 12 min. Find the when the temperature will be reduced to 40°C if the temperature of surrounding air is 30°C

Given that

$$\theta_1 = 120^{\circ}\text{C} \quad , \quad t = 0$$

$$\theta = 80^{\circ}\text{C} \quad t = 12$$

$$\theta = 40^{\circ}\text{C} \quad t = ? \quad , \quad \theta_0 = 30^{\circ}\text{C}$$

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\frac{d\theta}{\theta - 30} = -k dt$$

$$\int \frac{1}{\theta - 30} d\theta = -k \int dt + c$$

$$\log(\theta - 30) = -kt + c \quad \text{--- (1)}$$

$$\text{G.T. when } t = 0, \quad \theta = 120^{\circ}\text{C}$$

$$\log(120 - 30) = c$$

$$\log(90) = c$$

Substitute c in eqn (1)

$$\log(\theta - 30) = -kt + \log(90)$$

$$\log \frac{\theta - 30}{100} = -kt \quad \text{--- (2)}$$

Also Given $\theta = 80^\circ$, $t = 12$.

$$\log \frac{80 - 30}{100} = -k(12)$$

$$\log \frac{50}{100} = -12k$$

$$\log \frac{5}{10} = -12k$$

$$-k = \frac{1}{12} \log \frac{5}{9}$$

Sub $-k$ in (2),

$$\log \frac{\theta - 30}{100} = \frac{1}{12} \log \frac{5}{9} \times t \quad \text{--- (3)}$$

$$\log \frac{40 - 30}{100} = \frac{1}{12} \log \frac{5}{9} \times t$$

$$\log \frac{10}{100} = \frac{1}{12} \log \frac{5}{9}$$

$$\log \frac{1}{10} = \frac{1}{12} \log \frac{5}{9}$$

$$-t = \frac{12 \log (1/10)}{\log (5/9)}$$

- ④ A body kept in air with temperature 25°C cools from 140 to 80°C in 20 min . Find when the body cools down to 35°C .

Law of natural growth & decay:-

The rate of change of a quantity at time 't' is proportional to the quantity present at that time.

That is the differential equation is given by $\frac{dN}{dt} \propto N$

If t increases and N increases

then $\frac{dN}{dt} = kN$ (for growth)

If t increases and N decreases

then $\frac{dN}{dt} = -kN$ (for decay)

- ① Bacteria in a culture grow exponentially. So, the initial number is doubled in 3 hrs. How many times, the initial number will be present after 9 hrs.

Let N_0 be the initial number.

when $t=0$, $N=N_0$

$t=3$, $N=2N_0$

$t=9$, $N=?$

By law of natural growth

$$\frac{dN}{dt} = kN$$

$$\Rightarrow \frac{dN}{N} = k dt$$

on integration

$$\log N = kt + C \quad \text{--- (1)}$$

when $t=0$, $N=N_0$

$$\log N_0 = C$$

Sub 'c' in ①.

$$\log N = kt + \log N_0.$$

$$\log N - \log N_0 = kt$$

$$\log \frac{N}{N_0} = kt \quad \text{--- ②}$$

when $t = 3$, $N = 2N_0$.

$$\therefore \text{②} \Rightarrow \log \frac{2N_0}{N_0} = 3k$$

$$k = \frac{1}{3} \log 2$$

Sub 'k' in eqⁿ ②.

$$\log \frac{N}{N_0} = \frac{1}{3} (\log 2) t \quad \text{--- ③}$$

when $t = 9$, $N = ?$

$$\log \frac{N}{N_0} = \frac{1}{3} (\log 2) 9$$

$$\log \frac{N}{N_0} = 3 \log 2$$

$$\log \frac{N}{N_0} = \log 2^3$$

$$\frac{N}{N_0} = 8$$

$$N = 8N_0$$

After 9 hrs the initial number increases to $8N_0$.

- ⑤ The number N of bacteria in a culture grows at a rate proportional to N . The value of N was initially 100. Increased to 332 in 1 hr. What was the value of N after $1\frac{1}{2}$ hr.

Let N_0 be the initial number

when $t=0$, $N=100$

$t=60$ $N=332$

$t=90$ $N=?$

By law of natural growth.

$$\frac{dN}{dt} = kN$$

$$\Rightarrow \frac{dN}{N} = k dt$$

on Integrating

$$\log N = kt + c \quad \text{--- (1)}$$

when $t=0$, $N=100$

$$\textcircled{1} \Rightarrow \log 100 = c$$

$$\log N = kt + \log 100$$

$$\log N - \log 100 = kt$$

$$\log \frac{N}{100} = kt \quad \text{--- (2)}$$

when $t=60$, $N=332$

$$\textcircled{2} \Rightarrow \log \frac{332}{100} = k \cdot 60$$

$$\log 3.32 = 60k$$

$$k = \frac{1}{60} \log \frac{332}{100}$$

k value in eqn (2)

$$\log \frac{N}{100} = \left(\frac{1}{60} \log \frac{332}{100} \right) t$$

when $t = 90$, $N = ?$

$$\log \frac{N}{100} = \left(\frac{-1}{90} \log \frac{332}{100} \right) \frac{3}{2}$$

$$\log \frac{N}{100} = 3 \log \left(\frac{332}{100} \right)^{\frac{1}{2}}$$

$$\log \frac{N}{100} = 3 \log (11.0229)$$

$$\frac{N}{100} = \left(\frac{332}{100} \right)^{\frac{3}{2}}$$

$$\frac{N}{100} = (3.32)^{\frac{3}{2}}$$

$$\frac{N}{100} = 6.0493$$

$$N = 100 \times 6.0493$$

$$N = 604.93$$

$$N \approx 605$$

② If 30% of a radioactive substance disappears in 10 days how long will it take for 90% of it to disappear.

Let N_0 be initial amount

$$t=0, N=N_0$$

$$t=10, N=70\% (0.7)N_0$$

$$t=?, N=10\% (0.1)N_0$$

$$N=N_0 e^{-kt}$$

$$N = N_0 - 30\% N_0$$

$$N_0 = \frac{30}{100} N_0$$

$$\left(\frac{100-30}{100} \right) N_0$$

By law of natural decay

$$\frac{dN}{dt} = -kN$$

$$\frac{dN}{N} = -k dt$$

on integrating

$$\log N = -kt + c \quad \text{--- (1)}$$

when $t=0$, $N=N_0$

$$\log N_0 = C$$

Sub 'C' in eqⁿ ①

$$\log N = -kt + \log N_0$$

$$\log \frac{N}{N_0} = -kt \quad \text{--- ②}$$

when

$$t=10, \quad N=(0.7)N_0$$

$$\log \frac{(0.7)N_0}{N_0} = -k \cdot 10$$

$$\log (0.7) = -10k$$

$$-k = \frac{1}{10} \log (0.7)$$

∴ k. in value in ②

$$\log \frac{N}{N_0} = \frac{1}{10} \log (0.7) t$$

$$\text{when } t=? \quad N=(0.1)N_0$$

$$\log \frac{0.1 N_0}{N_0} = \frac{1}{10} \log (0.7) t$$

$$\log 0.1 = \frac{t}{10} \log 0.7$$

$$1 \log 0.1 = -t \log (0.7)$$

$$\frac{10 \log 0.1}{\log 0.7} = t$$

$$64.55 = t$$

$$65 \approx t$$

Higher order Linear differential Equations of Applications.

- 1) Homogeneous L.D.E with Constant Coefficients.
2. Non-homogeneous L.D.E with Constant Coefficients.
3. L.D.E with variable coefficients $\rightarrow$ Legendre's & Cauchy's
4. Method of variation of parameters.
5. Applications.

Homogeneous L.D.E with Constant Coefficients :-

- An equation of the form

$$a_0 \frac{d^2 y}{dx^2} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = 0 \quad \text{--- (1)}$$

where

$a_0, a_1, a_2, \dots, a_n$ are constants.

- Then equ (1) is called homogeneous L.D.E with Constant coefficients.

Let us denote

$$\frac{d}{dx} = D, \quad \frac{d^2}{dx^2} = D^2, \quad \frac{d^3}{dx^3} = D^3, \dots, \quad \frac{d^n}{dx^n} = D^n$$

Now eqⁿ (1) can be written as:

$$[a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n] y = 0$$

$$\Rightarrow f(D) y = 0 \quad \text{--- (2)}$$

where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n$

is a polynomial in D of degree n .

General solution of homogeneous

G.S of eqⁿ (2) is obtained by replacing D by m in $f(D)$, and solve the algebraic equation $f(m)=0$ [It contains n roots] $m_1, m_2, m_3, \dots, m_n$

The G.S is depending on the nature of the roots of an algebraic equation $f(m)=0$

Case 1:- Roots are real and distinct

Suppose m_1, m_2 are two real roots and distinct ($m_1 \neq m_2$)

Then the G.S can be written as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

In general $m_1, m_2, m_3, \dots, m_n$ are all real and distinct

Then the G.S can be written as

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

Case 2:- Roots are Real and equal.

Suppose $m_1 = m_2$ (that is first two roots are equal) and remain $n-2$ roots $m_3, m_4, \dots, m_n$ are distinct

Then the G.S can be written as

$$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$$

In general if $m_1 = m_2 = m_3 = \dots = m_r$, ($m_{r+1}, m_{r+2}, \dots, m_n$) and remain $(n-r)$ roots are distinct

Then the G.S. can be written as.

$$y = [c_1 + c_2x + c_3x^2 + c_4x^3 + \dots + c_r x^{r-1}] e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + c_{r+2} e^{m_{r+2} x} + \dots + c_n e^{m_n x}$$

Case (iii) Roots are Real & distinct

Suppose $\alpha + i\beta$ is one root then its complex conjugate $\alpha - i\beta$ is also a root

Suppose $m_1 = \alpha + i\beta$

then $m_2 = \alpha - i\beta$ are two complex and distinct roots, and remaining $n-2$ roots are all real and distinct

Then G.S. can be written as

$$y = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x] + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$$

Case (iv) Complex & Equal.

Suppose $m_1 = m_2 = \alpha + i\beta$. $\alpha + i\beta$ is a complex root repeated 2 times then its complex conjugate $\alpha - i\beta$ is also a root which is also repeated 2 times that is $m_3 = m_4 = \alpha - i\beta$ and remain $n-4$ roots are real and distinct.

Then G.S. can be written as.

$$y = e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \dots + c_n e^{m_n x}$$

Solve

$$\textcircled{1} \quad 2 \frac{d^2 y}{dx^2} - 13 \frac{dy}{dx} + 15y = 0$$

Given that

$$2 \frac{d^2 y}{dx^2} - 13 \frac{dy}{dx} + 15y = 0$$

Operator form is

$$[2D^2 - 13D + 15]y = 0$$

$$\Rightarrow -f(D)y = 0$$

$$\text{where } f(D) = 2D^2 - 13D + 15$$

Consider A.E. is $f(m) = 0$

$$\Rightarrow 2m^2 - 13m + 15 = 0$$

$$2m^2 - 10m - 3m + 15 = 0$$

$$2m[m-5] - 3[m-5] = 0$$

$$(2m-3)(m-5) = 0$$

$$m = \frac{3}{2}, m = 5, \text{ roots are real}$$

and distinct.

Then G.S. is

$$y = c_1 e^{\frac{3}{2}x} + c_2 e^{5x}$$

$$\text{or } y = c_1 e^{5x} + c_2 e^{\frac{3}{2}x}$$

$$\textcircled{2} \quad 4 \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + y = 0$$

G.T

$$4 \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + y = 0$$

Operator form is.

$$[4D^2 - 4D + 1]y = 0$$

$$\Rightarrow f(D)y = 0$$

$$\text{where } f(D) = 4D^2 - 4D + 1$$

$$\text{Consider } \text{A.E. ps. } f(m) = 0$$

$$\Rightarrow 4m^2 - 4m + 1 = 0$$

$$4m^2 - 2m - 2m + 1 = 0$$

$$2m[2m-1] - 1[2m-1] = 0$$

$$[2m-1][2m-1] = 0$$

$$m = \frac{1}{2}, m = \frac{1}{2} \quad \text{Roots are real and equal.}$$

G.S. ps.

$$y = (C_1 + C_2 x)e^{\frac{1}{2}x}$$

② $[D^2 - (a+b)D + ab]y = 0$, a and b being real & unequal.

④ $(D^3 + 3D^2 - 4)y = 0$

⑤ $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$

⑥ $(D^2 - 6D + 13)y = 0$

⑦ $(D^3 + 16D)y = 0$

⑧ $y'' - 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$.

⑨ $(D^3 + 3D^2 - 4)y = 0$

Given that

$$(D^3 + 3D^2 - 4)y = 0$$

$$f(D)y = 0$$

$$\text{where } f(D) = D^3 + 3D^2 - 4$$

Consider $\Delta \in \mathbb{R}$ $P_n - f(m) = 0$.

$$\Rightarrow m^3 + 3m^2 - 4 = 0.$$

$$m^3 + 3m^2 = 4$$

$$m(m^2 + 3m) = 4$$

$$m^3 + 3m^2 - 4 = 0.$$

$$(m-1)(m+2)(m+2) = 0.$$

$$m = 1, -2, -2.$$

All the roots are real but two roots are equal.

$$y = c_1 e^x + (c_2 + c_3 x) e^{-2x}$$

⑤ $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 5y = 0$

Given that

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 5y = 0$$

Operator form is

$$[D^2 - 4D + 5]y = 0$$

$$\Rightarrow f(D)y = 0$$

$$\text{where } f(D) = D^2 - 4D + 5$$

Consider $\Delta \in \mathbb{R}$ $f(m) = 0$
 $D^2 - 4D + 5 = 0$

$$\Rightarrow m^2 - 4m + 5 = 0.$$

$$= 7m^2 - 4m + 5 = 0$$

$$(m^2 - 5m + m + 5 = 0$$

$$m(m-5) + 1(m+5)$$

$$= m^2 + m + 5m + 5 = 0$$

$$\Rightarrow m(m-1) - 5(m-1) = 0$$

$$(m-1)(m-5) = 0$$

$m=1$, $m=5$ roots are real

and distinct. $m = \frac{4 \pm \sqrt{16-20}}{2}$

Then G.S is,

$$= \frac{4 \pm 2i}{2} = 2 \pm i$$

$$y = C_1 e^x + C_2 e^{5x}$$

$$\alpha \pm i\beta$$

$$\alpha=2, \beta=1$$

$$y = C_1 e^{5x} + C_2 e^x$$

$$y = e^{2x} [C_1 \cos x + C_2 \sin x]$$

(B) $(D^2 - 6D + 13)y = 0$

Given that

$$(D^2 - 6D + 13)y = 0 \quad \text{Operator form}$$

$$(D^2 - 6D + 13)y = 0$$

$$\Rightarrow f(D)y = 0$$

$$\text{where } f(D) = D^2 - 6D + 13$$

$$\text{Consider } A \in \mathbb{R} \text{ s.t. } f(m) = 0$$

$$D^2 - 6D + 13 = 0$$

$$\Rightarrow m^2 - 6m + 13 = 0$$

$$a=1, b=-6, c=13$$

$$D_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a_1 = \frac{6 \pm \sqrt{36 - 4(1)(3)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36 - 12}}{2}$$

$$= \frac{6 \pm \sqrt{24}}{2}$$

$$= \frac{6 \pm \sqrt{16}i}{2}$$

$$= \frac{6 \pm 4i}{2}$$

$$m_1 = \frac{6+4i}{2}, \quad m_2 = \frac{6-4i}{2}$$

are real & distinct.

Then G.S. is

$$y = e^{3x} \left[C_1 \cos \frac{4}{2}x + C_2 \sin \frac{4}{2}x \right]$$

$$y = e^{3x} \left[C_1 \cos 2x + C_2 \sin 2x \right]$$

Non homogeneous L.D.E

An ordinary D.E of the form.

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = Q(x) \quad (1)$$

where $a_0, a_1, a_2, \dots, a_n$ are constant and $Q(x)$ is a continuous function of x is known as non homogeneous linear D.E of order n with constant coefficients. eqn (1) can be also be written as

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = Q(x)$$

$$= \mathcal{F}(D) y = Q(x)$$

$$\text{where } \mathcal{F}(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n$$

General solution of non-homogeneous L.D.E

A general solution of

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = Q(x) \quad (1)$$

is obtained in two parts as

G.S = Complementary function + Particular Integral.

$$\boxed{y = C.F + P.I}$$

Complementary function (C.F) is the general solution of the homogeneous equation obtained by putting $Q(x) = 0$ in (1)

Particular Integral (P.I) is a particular solution of the nonhomogeneous equation (1) and contains no arbitrary constants

Inverse operator and particular integral.

Since $\mathcal{F}(D)$ is known as differential operator and $\frac{1}{\mathcal{F}(D)}$ is known as Inverse differential operator.

The particular Integral (P.I) of the equation $\mathcal{F}(D)y = Q(x)$

$$\mathcal{F}(D)y = Q(x) \text{ is } \frac{1}{\mathcal{F}(D)} Q(x)$$

i.e. P.I = $\frac{1}{\mathcal{F}(D)} Q(x)$ and it does not contain any arbitrary constants

Ans: -

$$n \text{ of } (y - ay) = q(x) \text{ is } e^{ax} / f(a) e^{ax} dx$$

$$\text{ie } P.I. = \frac{1}{a-a} q(x) e^{ax} = \frac{1}{f(a)} e^{ax} dx$$

Some methods of obtaining particular integral

The method depends on the nature of $q(x)$

$$\text{eg } f(y) = q(x)$$

P.I. by this method can be obtained when $q(x)$ has the following form:

$$I, q(x) = e^{ax}$$

$$II, q(x) = \sin ax \text{ or } \cos ax$$

$$III, q(x) = x^n \text{ or polynomial in } x$$

$$IV, q(x) = e^{ax} v(x) \text{ where } v(x) = \sin ax \text{ or } \cos ax \text{ or } x^n$$

$$V, q(x) = x^n v(x) \text{ where } v(x) = \sin ax \text{ or } \cos ax$$

Method - I: When $q(x) = e^{ax}$

$$P.I. = \frac{1}{f(a)} q(x) = \frac{1}{f(a)} e^{ax} \text{ if } f(a) \neq 0$$

If $f(a) = 0$ then

$$P.I. = \frac{1}{f(a)} e^{ax} = \frac{x}{f'(a)} e^{ax} \text{ where } f'(a) \neq 0$$

If $f'(a) = 0$ then

$$P.I. = \frac{1}{f(a)} e^{ax} = \frac{x^2}{f''(a)} e^{ax} \text{ where } f''(a) \neq 0$$

In general, if $f(a) = 0$ then

$$P.I. = \frac{1}{f(a)} e^{ax} = \frac{x^{r+1}}{f^{(r+1)}(a)} e^{ax} \text{ where } f^{(r+1)}(a) \neq 0$$

For this

$$\text{solve } (D^2 + D - 4)y = e^{2x}$$

Given that

$$(D^2 + D - 4)y = e^{2x}$$

$$-f(x)y = e^{2x}$$

$$-f(x) D^2 + D - 4 \text{ is } f(x) = e^{2x} (2x+3)$$

To find C.F. :-

$$\text{Consider } m \in \mathbb{R} \quad -f(m) = 0$$

$$\Rightarrow m^2 + m - 4 = 0$$

$$\Rightarrow m^2 + 3m - 2m - 6 = 0$$

$$\Rightarrow m(m+3) - 2(m+3) = 0$$

$$(m-2)(m+3) = 0$$

$m = 2, -3$ roots are distinct

C.F. of (1) :-

$$y_c = c_1 e^{2x} + c_2 e^{-3x}$$

To find P.I. :-

$$y_p = \frac{1}{f(x)} Q(x)$$

$$= \frac{1}{f(x)} e^{2x} \quad (a=2)$$

$$\therefore f(2) = 0$$

$$y_p = \frac{x}{f'(2)} e^{2x}$$

$$f(x) = D^2 + D - 4$$

$$f(2) = 0$$

$$f'(x) = 2x + 1$$

$$= 5/2$$

$$\therefore y_p = \frac{x}{5} e^{2x}$$

$$\text{G.S.P.S } y = C_1 e^{2x} + C_2 e^{-3x} + \frac{x}{5} e^{2x}$$

Q. solve $(D^3 - D^2 + 4D - 4)y = e^x$

Given that

$$(D^3 - D^2 + 4D - 4)y = e^x$$

$$f(D)y = Q(x)$$

where

$$f(D) = D^3 - D^2 + 4D - 4 \text{ and } Q(x) = e^x$$

To find C.F. is

$$-f(D) = D^3 - D^2 + 4D - 4$$

$$-A \text{ or } B - f(m) = 0$$

$$\rightarrow m^3 - m^2 + 4m - 4 = 0$$

$$(m-1)(m^2+4) = 0$$

$$m-1=0 \quad m^2+4=0$$

$$m=1 \quad m = \pm 2i \quad (\alpha \pm i\beta)$$

$$(\alpha=0, \beta=2)$$

Out of three roots one is real & two roots are complex distinct

$\therefore$ C.F.

$$y_c = C_1 e^x + e^{0x} [C_2 \cos 2x + C_3 \sin 2x]$$

$$y_c = C_1 e^x + C_2 \cos 2x + C_3 \sin 2x \quad \text{--- (2)}$$

$$P.I = y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} e^x \quad (C=1)$$

$$\therefore f(1) = 0$$

$$\therefore y_p = \frac{x}{f'(1)} e^x$$

$$\therefore f(D) = D^3 - D^2 + 4D - 4$$

$$f'(D) = 3D^2 - 2D + 4$$

$$f'(1) = 3(1)^2 - 2(1) + 4$$

$$1 + 4$$

$$f'(1) = 5 \neq 0$$

$$y_p = \frac{x}{5} e^x$$

$$\text{G. S. P. S. } y = C_1 e^x + C_2 \cos 2x + C_3 \sin 2x + \frac{x}{5} e^x$$

3. solve $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 3e^{-x} + 2$

Given

$$(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 3e^{-x} + 2$$

$$f(D)y = Q(x)$$

where

$$f(D) = D^3 - 5D^2 + 8D - 4 \quad \& \quad Q(x) = e^{2x} + 3e^{-x} + 2$$

To find C.F. k

$$f(D) = D^3 - 5D^2 + 8D - 4$$

$$A.E. \text{ P.S. } f(m) = 0$$

$$m^3 - 5m^2 + 8m - 4 = 0$$

$m = 2, 2, 1$ are real & two roots are equal

C.F. is

$$y_c = (C_1 + C_2 x) e^{2x} + C_3 e^x$$

To find P.I. :-

$$y_p = \frac{1}{f(D)} Q(x)$$

$$\begin{array}{c|cccc} 0 & 1 & -5 & 8 & -4 \\ & 0 & 2 & -6 & 4 \\ \hline 1 & 1 & -3 & 2 & 0 \\ & 0 & 1 & -2 & 0 \\ \hline 2 & 1 & -2 & 0 & 0 \\ & 0 & 2 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 \end{array}$$

$$= \frac{1}{-f(0)} [e^{2x} + 3e^{-x} + 2]$$

$$= \frac{1}{-f(0)} e^{2x} + \frac{1}{-f(0)} 3e^{-x} + \frac{1}{-f(0)} 2$$

$$= y_{P1} + y_{P2} + y_{P3} \quad (3)$$

$$\text{Consider } y_{P1} = \frac{1}{f(0)} e^{2x} \quad (a=2)$$

$$-f(2) = 0 \text{ \& } f'(2) = 0$$

$$\therefore y_{P2} = \frac{x^2}{f''(0)} e^{2x}$$

$$\therefore f(D) = D^3 - 5D^2 + 8D - 4$$

$$f'(D) = 3D^2 - 10D + 8$$

$$f''(D) = 6D - 10$$

$$f''(2) = 6(2) - 10 \\ = 2, \neq 0.$$

$$y_{P1} = \frac{x^2}{2} e^{2x} \quad (a)$$

$$y_{P2} = \frac{x^2}{2} 3e^{-2x} \quad (a = -1)$$

$$y_{P3} = \frac{x^2}{2} 2$$

$$\therefore f(-1) \neq 0$$

$$-f(0) = D^3 - 5D^2 + 8D - 4$$

$$-f(-1) = (-1)^3 - 5(-1)^2 + 8(-1) - 4$$

$$= -1 - 5 - 8 - 4$$

$$= -18 \neq 0$$

$$\therefore y_{P2} = \frac{1}{-f(-1)} 8e^{-x} = \frac{8}{18} e^{-x} = \frac{-1}{6} e^{-x} \quad (b)$$

$$y_{p3} = \frac{1}{f(D)} 2e^{0x} \quad (a=0).$$

$$= \frac{1}{f(0)} 2(1)$$

$$= \frac{1}{-4} 2$$

$$= -\frac{1}{2} - 3(0).$$

Substitute $3a, 3b, 3c$ in 3

$$= \frac{x^2}{2} 3e^{-2x} + \frac{1}{6} e^{-x} - \frac{1}{2}$$

General solution.

$$y = y_c + y_p.$$

$$= (C_1 + C_2 x) e^{2x} + C_3 e^x + \frac{x^2}{2} 3e^{-2x} - \frac{1}{6} e^{-x} - \frac{1}{2}$$

$$\text{Solve } (D^2 - 2D + 1)y = e^x + 1$$

$$\text{Solve } (D^2 - 9)y = e^{-3x} + 1 + e^{3x}$$

$$\text{Solve } (D^2 - D - 6)y = e^x \cosh 2x$$

$$\text{Solve } \frac{d^3 y}{dx^3} + y = (e^x + 1)^2$$

Method II :- When $Q(x) = \sin ax$ or $\cos ax$

If $Q(x) = \sin ax$ then

$$P.I = \frac{1}{f(D)} \sin ax$$

$$= \frac{1}{f(D^2)} \sin ax \quad (\text{consider polynomial in } D^2 \text{ only})$$

$$= \frac{1}{f(-a^2)} \sin ax \quad \text{if } f(-a^2) \neq 0$$

Suppose $f(-a^2) = 0$ then.

$$P.I = \frac{x}{f'(-a^2)} \sin ax \quad \text{where } f'(-a^2) \neq 0$$

$$\text{If } f'(-a^2) = 0$$

$$\text{then P.I.} = \frac{x^2}{f''(-a^2)} \sin ax \text{ where } f''(-a^2) \neq 0$$

In general, If $f'(-a^2) = 0$ then

$$\text{P.I.} = \frac{1}{f(0)} e^{ax}$$

$$= \frac{x^{n+1}}{f^{(n+1)}(a)} e^{ax} \text{ where } f^{(n+1)}(a) \neq 0$$

Problems Method - I

1. Solve $(D^2 + D - 6)y = e^{2x}$
2. Solve $(D^3 - D^2 + 4D - 4)y = e^x$
3. Solve $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 3x^{-x} + 2$
4. Solve $(D^2 - 2D + 1)y = e^x + 1$
5. Solve $(D^2 - 9)y = e^{-3x} + 1 + e^{3x}$
6. Solve $(D^2 - D - 6)y = e^x \cosh 2x$
7. Solve $\frac{d^3 y}{dx^3} + y = (e^x + 1)^2$

NOTE I =

If after replacing D^2 by $-a^2$, $f(D)$ contains terms D , then denominator is rationalized to obtain the even power of D .

Problem 1 = method - II

1. Solve $(D^2 + 9)y = \sin 4x$
2. Solve $(D^4 + 8a^2 D^2 + a^4)y = 8 \cos ax$
3. Solve $(D^2 + 3D + 2)y = \sin 2x$
4. Solve $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$
5. Solve $(D^2 + 9)y = \cos^2 x$

6. solve $(D^2 - 8D + 9)y = 5\sin 5x$
 7. solve $(D^4 + 10D^2 + 9)y = \cos(2x + 3)$
 8. solve $(D^2 - 20D + 9)y = e^x + \cos x$
 9. solve $(D^2 - 4D + 13)y = 5\sin 3x$, $y(0) = 1$, $y'(0) = 2$

① $(D^2 + 9)y = \sin 4x$

Given that

$$(D^2 + 9)y = \sin 4x$$

$$f(D)y = Q(x)$$

$$f(D) = D^2 + 9$$

$$Q(x) = \sin 4x$$

To find C.F. =

where

$$f(m) = 0$$

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$m = \pm 3i$$

$$m = \pm 3i$$

$$(\alpha \pm i\beta)$$

$$\alpha = 0$$

$$\beta = 3$$

Roots are complex and distinct

$$y_c = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

$$y_c = C_1 \cos 3x + C_2 \sin 3x$$

P.I. :-

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2 + 9} \sin 4x \quad (\alpha = 4)$$

$$[- \because f(4^2) \neq 0]$$

put $D^2 = -4^2 = -16$

$$y_p = \frac{1}{-16 + 9} \sin 4x$$

$$= \frac{1}{-17} \sin 4x \text{ --- (3)}$$

G.S is

$$y = y_c + y_p$$

$$y = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{17} \sin 4x$$

$$\textcircled{2} (D^4 - 2a^2 D^2 + a^4) y = 8 \cos ax$$

G.T.

$$(D^4 + 2a^2 D^2 + a^4) y = 8 \cos ax$$

$$-f(D) = D^4 + 2a^2 D^2 + a^4$$

$$\phi(x) = 8 \cos ax$$

To find C.F. =

where

$$-f(m) = 0$$

$$-f(m) = D^4 + 2a^2 D^2 + a^4$$

$$(m^2 + a^2)^2 = 0$$

$$m^2 + a^2 = 0$$

$$m^2 = -a^2$$

$$m = \pm ai \text{ --- twice.}$$

$$(x \pm i\beta) \text{ where } \beta = a$$

Roots are complex and equal

$$y_c = e^{ax} [(c_1 + c_2 x) \cos ax + (c_3 + c_4 x) \sin ax]$$

$$y_c = (c_1 + c_2 x) \cos ax + (c_3 + c_4 x) \sin ax \text{ --- (2)}$$

To find P.I. =

$$y_p = \frac{1}{-f(D)} \phi(x)$$

$$= \frac{1}{-f(D)} 8 \cos ax$$

$$\therefore f(-a^2) = 0 \quad f'(-a^2) = 0$$

$$[\because f'(D) = 4D^3 + 4a^2D]$$

$$y_p = \frac{x^2}{f''(-a^2)} 8 \cos ax$$

$$f''(D) = 12D^2 + 4a^2$$

$$f(-a^2) = -8a^2]$$

$$= \frac{x^2}{-8a^2} 8 \cos ax$$

$$= -\frac{x^2}{a^2} \cos ax \quad \text{--- (3)}$$

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x) \cos ax + (c_3 + c_4 x) \sin ax - \frac{x^2}{a^2} \cos ax.$$

$$(3) (D^2 + 3D + 2)y = \sin x.$$

G.T

$$(D^2 + 3D + 2)y = \sin x$$

$$-f(D) = D^2 + 3D + 2$$

$$Q(x) = \sin x$$

To find C.I.

$$-f(m) = 0$$

$$m^2 + 3m + 2 = 0$$

$$m^2 + 2m + m + 2 = 0$$

$$m(m+2) + 1(m+2) = 0$$

$$m = -2 \quad m = -1$$

$$m = -1, -2$$

Roots are real and distinct

$$y_c = c_1 e^{-x} + c_2 e^{-2x} \quad \text{--- (2)}$$

To find P.I. :-

$$y_p = \frac{1}{-f(D)} Q(x)$$

$$-f(D) = m^2 + 3m^2 + 2$$

$$Q(x) = \sin 2x$$

$$y_p = \frac{1}{-f(D)} \sin 2x \quad (a=2)$$

$$\text{put } D^2 = -a^2$$

$$= -4$$

$$y_p = \frac{1}{-4 + 3D + 2} \sin 2x$$

$$y_p = \frac{1}{3D - 2} \sin 2x$$

$$= \frac{3D + 2}{9D^2 - 4} \sin 2x$$

$$\text{put } D^2 = -2^2$$

$$y_p = \frac{3D + 2}{9(-4) - 4} \sin 2x$$

$$= \frac{3D + 2}{-40} \sin 2x$$

$$= \frac{-1}{40} 3D \sin 2x + 2 \sin 2x$$

$$= \frac{-1}{40} 6 \cos 2x + 2 \sin 2x$$

$$y_p = \frac{-1}{20} 3 \cos 2x + \sin 2x$$

$$y = y_p + y_c$$

$$y = c_1 e^x + c_2 e^{-2x} - \frac{1}{20} 3 \cos 2x + \sin 2x$$

Method II:-

When $Q(x) = x^m$ or polynomial in x where m is positive integer.

$$\begin{aligned}\text{We have P.I} &= \frac{1}{-f(D)} Q(x) \\ &= \frac{1}{-f(D)} x^m\end{aligned}$$

To evaluate P.I reduce $\frac{1}{-f(D)}$ to the form

$\frac{1}{1 \pm \phi(D)}$ by taking out the lowest degree terms from $-f(D)$

Now write $\frac{1}{-f(D)}$ as $[1 \pm \phi(D)]^{-1}$ and

expand it in ascending powers of D using Binomial theorem up to the term containing D^m .

Then operate x^m with the terms of the expansion of $[1 \pm \phi(D)]^{-1}$

NOTE:-

$$1) \frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$$

$$2) \frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$$

$$3) \frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \dots$$

$$4) \frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \dots$$

$$5) \frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \dots$$

$$6) \frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \dots$$

Problem :-

1) Solve $(D^2 + 2D + 1)y = x$

2) Solve $(D^2 - 4)y = x^2$

3) Solve $(D^3 - D^2 - D + 1)y = 1 + x^2$

4) Solve $(D^2 + 2)y = x^3 + x^2 + e^{-2x} + \cos 3x$

5) Solve $(D^2 + D + 1)y = x^3$

6) Solve $(D^2 + 2D + D)y = 2x + x^2$

Method IV :- When $Q(x) = e^{ax} v(x)$ where $v(x) = \sin ax$ or $\cos ax$ or x^m by definition, we have

$$P.I = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} e^{ax} v(x)$$

To find P.I take e^{ax} left side of $\frac{1}{f(D)}$ and replace every D with $D+a$ in $f(D)$ and operate it with $v(x)$ using previous methods

$$P.e., P.I = \frac{1}{f(D)} e^{ax} v(x)$$

$$= e^{ax} \frac{1}{f(D+a)} v(x)$$

Problem :-

1. Solve $(D^2 + 2)y = e^x \cos x$

2. Solve $(D^3 - 3D^2 + 3D - 1)y = x^2 e^x$

3. Solve $(D^2 + 4D + 3)y = e^{-x} \sin x + x$

4. Solve $(D^2 - 4D + 3)y = 2xe^{3x} + 3e^x \cos 2x$

5. Solve $(D^2 + 4)y = e^x \sin^2 x$

6. Solve $(D^4 - 1)y = \cos x \cosh x$

7. Solve $(D^3 - 2D + 4)y = e^x \sin x$

8. Solve $(D^2 - 2D + 1)y = x^2 e^{3x}$

Method - II

① solve $(D^2 + 2D + 1)y = x$

Given that $(D^2 + 2D + 1)y = x$

$$f(D)y = Q(x)$$

To find C.F.:

$$A.E. \text{ is } f(m)$$

$$\Rightarrow m^2 + 2m + 1 = 0$$

$$(m+1)^2$$

$m = -1, -1$ are real & equal.

$$y_c = (C_1 + C_2 x)e^{-x}$$

To find P.I.:

$$y_p = \frac{1}{f(D)} Q(x)$$

$$y_p = \frac{1}{D^2 + 2D + 1} x \quad (m = 1)$$

$$= \frac{1}{(1+D)^2} x$$

$$= (1+D)^{-2} x$$

$$= (1 - 2D + 3D^2 - 4D^3 + \dots) x$$

$$= x - 2D(x) + 3D^2(x) + \dots$$

$$= x - 2(1) + \dots = 0$$

$$= x - 2$$

② solve $D^2 - 4$

General solution

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x) e^{-x} + x - 2$$

③ Solve $(D^2 - 4)y = x^2$

Given that

$$(D^2 - 4)y = x^2 \quad \text{--- (1)}$$

$$-f(D)y = Q(x)$$

To find C.F. :-

$$-A \in P_s \quad f(m) = 0$$

$$D^2 - 4 = 0$$

$$D^2 = 4$$

$D = 2, -2$ are real and equal.

$$y_c = C_1 e^{2x} + C_2 e^{-2x}$$

To find P.I. :-

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2 - 4} x^2 \quad (m=2)$$

$$= \frac{1}{-4(1 - \frac{D^2}{4})} \cdot x^2$$

$$= \frac{-1}{4} \left(1 - \frac{p^2}{4}\right)^{-1} x^2$$

$$= \frac{-1}{4} \left[1 + \frac{p^2}{4} + \left(\frac{p^2}{4}\right)^2 \right] x^2$$

$$= \frac{-1}{4} \left[x^2 + \frac{p^2(x^2)}{4} + \frac{p^4(x^2)}{16} \right]$$

$$= \frac{x^2}{4} - \frac{1}{4} \left[x^2 + \frac{1}{4}(2) + \frac{1}{16}(0) \right]$$

$$y_p = \frac{-1}{4} \left[x^2 + \frac{1}{2} \right]$$

General solution.

$$y = y_c + y_p$$

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{x^2}{4} - \frac{1}{8}$$

⑧ solve $(D^3 - D^2 - D + 1)y = 1 + x^2$

$$(D^3 - D^2 - D + 1)y = 1 + x^2 \quad \text{--- ①}$$

is in the form of $f(D)y = Q(x)$

To find R.F. =

$$f(m) = 0$$

$$m^3 - m^2 - m + 1 = 0$$

$$m = 1, 1, -1$$

$\therefore$ two roots are and one root is distinct
and all roots are real.

Then

$$y_c = (c_1 + c_2 x)e^x + c_3 e^{-x}$$

To find P.I.:-

$$y_p = \frac{1}{f(D)} Q(x)$$

$$y_p = \frac{1}{D^3 - D^2 - D + 1} (1+x^2)$$

$$= \frac{1}{(1 + (D^3 - D^2 - D))} (1+x^2)$$

$$= (1 + (D^3 - D^2 - D))^{-1} (1+x^2)$$

$$= 1 + (D^3 - D^2 - D)$$

$$= [1 + (D^3 - D^2 - D) + (D^3 - D^2 - D)^2] (1+x^2)$$

$$= (1 + D^2 + D + D^2) (1+x^2) \quad (\text{neglect higher order derivatives more than 2})$$

$$= (1 + D + 2D^2) (1+x^2)$$

$$= (1+x^2) + D(1+x^2) + 2D^2(1+x^2)$$

$$= (1+x^2 + 2x + 2x^2)$$

$$= 1 + x^2 + 2x + 2x^2$$

$$= x^2 + 2x + 1$$

G.S

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x)e^x + c_3 e^{-x} + x^2 + 2x + 1$$

Q) $(D^2+2)y = x^2 + x + e^{-2x} + \cos 3x$
 is form of $f(D)y = Q(x)$

To find C.F.I.

$$f(m) = 0$$

$$(m^2 + 2 = 0)$$

$$m^2 = -2$$

$$m = \pm \sqrt{-2}$$

$$m = \pm \sqrt{2}i \quad (\alpha + i\beta)$$

$$\alpha = 0$$

$$\beta = \sqrt{2}$$

Roots are complex & Distinct.

$$y_c = e^{0x} (C_1 \cos \sqrt{2}x + C_2 \sin \sqrt{2}x)$$

$$= C_1 \cos \sqrt{2}x + C_2 \sin \sqrt{2}x$$

To find P.I. :-

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} \left[(x^2 + x^2) + e^{-2x} + \cos 3x \right]$$

$$= \frac{1}{f(D)} (x^2 + x^2) + \frac{1}{f(D)} e^{-2x} + \frac{1}{f(D)} \cos 3x$$

$$= y_{p1} + y_{p2} + y_{p3}$$

$$y_{p1} = \frac{1}{f(D)} (x^2 + x^2)$$

$$= \frac{1}{(D^2 + 2)} x^2 + x^2 \quad (m=0)$$

$$= \frac{1}{2(1 + \frac{D^2}{2})} x^2 + x^2 \quad (m=0)$$

$$= \frac{1}{2} \left(1 - \frac{D^2}{2} + \left(\frac{D^2}{2} \right)^2 + \left(\frac{D^2}{3} \right)^3 \right) (x^3 + x^2)$$

$$= \frac{1}{2} \left(1 - \frac{D^2}{2} \right) (x^3 + x^2)$$

$$= \frac{1}{2} \left(x^3 + x^2 - \frac{D^2}{2}(x^3) - \frac{D^2}{2}(x^2) \right)$$

$$= \frac{1}{2} \left(x^3 + x^2 - \frac{1}{2} \cdot 6x - \frac{1}{2} \cdot (2) \right)$$

$$= \frac{1}{2} (x^3 + x^2 - 1) \frac{1}{2} x$$

$$y_{P1} = \frac{1}{2} (x^3 + x^2 - 1) - 3x$$

$$y_{P2} = \frac{1}{-f(0)} e^{-2x}$$

$$= \frac{1}{(D^2 + 2)} e^{-2x} \quad (a = -2)$$

$$= \frac{1}{-4} e^{-2x}$$

$$f(-2) \neq 0$$

$$= \frac{1}{-4} e^{-2x}$$

$$y_{P3} = \frac{1}{-f(0)} \cos 3x \quad (a = 3)$$

$$= \frac{1}{D^2 + 2} \cos 3x$$

$$\text{put } D^2 = -(3)^2 = -9$$

$$y_{P3} = \frac{1}{-9+2} \cos 3x$$

$$= \frac{1}{-7} \cos 3x - (3)$$

$$y_p = y_{p1} + y_{p2} + y_{p3}$$

$$= \frac{1}{2} (x^3 + x^2 - 1 - 3x) + \frac{1}{4} e^{-2x} - \frac{1}{7} \cos 3x$$

G.S.

$$y = y_c + y_p$$

$$y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x + \frac{1}{2} (x^3 + x^2 - 1 - 3x) - \frac{1}{4} e^{-2x} - \frac{1}{7} \cos 3x$$

5) solve $(D^2 + 0D + 1)y = x^3$

Given that

$$(D^2 + D + 1)y = x^3$$

$$f(D)y = Q(x)$$

To find C.F.I =

$$f(m) = 0$$

$$m^2 + m + 1 = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4}}{2}$$

$$m_1 = \frac{-1 + \sqrt{3}i}{2}, m_2 = \frac{-1 - \sqrt{3}i}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

Method - iv

$$D \text{ solve } (D^2 + 2)y = e^x \cos x.$$

Given that

$$(D^2 + 2)y = e^x \cos x$$

To find C.F. :-

$$f(D)y = Q(x)$$

$$f(m) = 0$$

$$m^2 + 2 = 0$$

$$m^2 = -2$$

$$m = \pm \sqrt{2}i$$

$$\alpha = 0$$

$$\beta = \sqrt{2}$$

— Here roots are equal and complex distinct

$$y_c = e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x]$$

$$y_c = C_1 \cos \sqrt{2} x + C_2 \sin \sqrt{2} x.$$

To find P.I. :-

$$P.I. = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{(D^2 + 2)} e^x \cos x$$

$$= e^x \frac{1}{(D^2 + 2)} \cos x$$

$$= e^x \frac{1}{(D + \alpha)^2 + 2} \cos x. \quad \alpha = 0$$

$$= e^x \frac{1}{(D+1)^2 + 2} \cos x$$

$$= e^x \frac{1}{(D^2 + 2D + 1) + 2} \cos x$$

$$= e^x \frac{1}{D^2 + 2D + 3} \cos x \quad (a=1)$$

$$= e^x \frac{1}{(-1^2) + 2D + 3} \cos x \quad \begin{matrix} D^2 = -1^2 \\ D^2 = -1 \end{matrix}$$

$$= e^x \frac{1}{2D + 2} \cos x$$

$$= e^x \frac{1}{2(D+1)} \cos x$$

$$= e^x \left(\frac{1}{2(D+1)} \right) \cos x$$

$$= \frac{1}{2} e^x \frac{D-1}{D^2-1} \cos x$$

$$= \frac{1}{2} e^x \frac{D-1}{-1-1} \cos x$$

$$= \frac{1}{2} e^x \frac{D-1}{-2} \cos x$$

$$= \frac{-1}{4} e^x D(\cos x) - \cos x$$

$$= \frac{-1}{4} e^x (-\sin x) - \cos x$$

$$y_p = \frac{+1}{4} e^x \sin x + \frac{1}{4} e^x \cos x$$

Q. 5

$$y = y_c + y_p$$

$$y = C_1 \cos \sqrt{2} x + C_2 \sin \sqrt{2} x + \frac{1}{4} e^x \sin x + \frac{1}{4} e^x \cos x$$

(2) Solve $(D^3 - 3D^2 + 3D - 1)y = x^2 e^x$.

Given that

$$(D^3 - 3D^2 + 3D - 1)y = x^2 e^x$$

To find C.F. :-

$$f(m) = 0$$

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$(m-1)(m^2 - 2m + 1) = 0$$

$$(m-1)(m^2 - m - m + 1) = 0$$

$$(m-1)m(m-1) - 1(m-1)$$

$$(m-1)(m-1)(m-1)$$

$$m = 1, 1, 1$$

roots are equal and real.

$$y_c = (C_1 + C_2 x + C_3 x^2) e^x$$

To find P.I. :-

$$P.I = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{(D-1)^3} x^2 e^x$$

$$= e^x \frac{1}{(0-1)^3} x^2 \quad a=1 \quad u(x)=x^2$$

$$= e^x \frac{1}{(0+1-1)^3} x^2$$

$$= e^x \frac{1}{0^3} x^2$$

$$= e^x \int \left[\int e^x dx \right] dx$$

$$= e^x \int \left[\int \frac{x^2}{3} dx \right] dx$$

$$D = \frac{d}{dx}$$

$$= \frac{e^x}{3} \int \frac{x^4}{4} dx$$

$$\frac{1}{D} = \int dx$$

$$= \frac{e^x}{4} \frac{x^5}{5}$$

$$= \frac{e^x x^5}{60}$$

$$y_p = \frac{e^x x^5}{60}$$

Ex. 5

$$y = y_c + y_p$$

$$y = (C_1 + C_2 x + C_3 x^2) e^x + \frac{e^x x^5}{60}$$

$$(D^2 + 4D + 3)y = e^{-x} \sin x + x$$

Given that

$$(D^2 + 4D + 3)y = e^{-x} \sin x + x$$

To find C.F. :-

$$f(m) = 0$$

$$m^2 + 4m + 3 = 0$$

$$m^2 + 3m + m + 3 = 0$$

$$m(m+3) + 1(m+3) = 0$$

$m = -1, -3$
roots are real and distinct

$$y_c = C_1 e^{-x} + C_2 e^{-3x}$$

To find P.I. :-

$$P.I. = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2 + 4D + 3} (e^{-x} \sin x + x)$$

$$= \left[\frac{e^{-x}}{D^2 + 4D + 3} \sin x + \frac{1}{D^2 + 4D + 3} x \right]$$

$$= \frac{1}{f(D)} e^{-x} \sin x + \frac{1}{f(D)} x$$

$$= y_{P1} + y_{P2}$$

$$y_{P1} = \frac{1}{D^2 + 4D + 3} e^{-x} \sin x$$

$$a = -1$$

$$= e^{-x} \frac{1}{(D^2 - 1)^2 + 4(D - 1) + 3} \sin x$$

$$= e^{-x} \frac{1}{D^2 + 1 + 2D + 4D - 4 + 3} \sin x$$

$$= e^{-x} \frac{1}{D^2 + 3D} \sin x$$

$$p^2 + 1 = -1 = -1$$

$$y_{p1} = e^{-x} \frac{1}{-1-20} \sin x$$

$$= e^{-x} \frac{(20+1)}{(20+1)(20-1)} \sin x$$

$$= e^{-x} \frac{20+1}{40^2-1} \sin x$$

$$p^2 + 1 = -1 = -1$$

$$= e^{-x} \frac{20+1}{4(-1)+1} \sin x$$

$$= e^{-x} \frac{20+1}{-5} \sin x$$

$$y_{p1} = \frac{e^{-x}}{5} (20 \cos x - 1 \sin x)$$

now

$$y_{p2} = \frac{1}{-f(0)} x$$

$$= \frac{1}{D^2 + 4D + 3} x$$

$$= \frac{1}{3 \left(1 + \left(\frac{D^2}{3} + \frac{4D}{3} \right) \right)} x$$

$$= \frac{1}{3} \left(1 - \left(\frac{D^2}{3} + \frac{4D}{3} \right) \right) x$$

$$y_{p2} = \frac{1}{3} \left(1 - \frac{D^2}{3} - \frac{4D}{3} \right) x$$

$$= \frac{1}{3} \left[x - 0 - \frac{4}{3} \right]$$

$$= \frac{x}{3} - \frac{4}{9}$$

$$y_p = y_{p1} + y_{p2}$$

$$y_p = -\frac{e^{-x}}{5} 2 \cos x + \sin x + \frac{x}{3} - \frac{4}{9}$$

$$y = y_c + y_p$$

$$y = c_1 e^{-x} + c_2 e^{-3x} + \frac{e^{-x}}{5} 2 \cos x + \sin x + \frac{x}{3} - \frac{4}{9}$$

Method II :- When $Q(x) = x^m v(x)$ where $v(x) = \sin ax$ or $\cos ax$

Case (i) :-

If $m=1$

$$\text{Then P.I.} = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} x v(x)$$

$$= \left[x - \frac{f'(D)}{f(D)} \right] \frac{1}{f(D)} v(x)$$

Case (ii)

If $m \neq 1$ and $v(x) = \sin ax$

$$\text{Then P.I.} = \frac{1}{f(D)} x^m \sin ax$$

$$= \frac{1}{f(D)} x^m \text{ (Imaginary part (I.P) of } e^{iax})$$

Similarly

$$\text{P.I.} = \frac{1}{f(D)} x^m \cos ax$$

$$= \text{R.P of } \frac{1}{f(D)} x^m e^{iax}$$

Problems:-

1. Solve $(D^2+4)y = \sin x$
2. Solve $(D^2-2D+1)y = xe^x \sin x$
3. Solve $(D^2-D)y = x^2 \cos x$
4. Solve $(D^2+4)y = \tan 2x$
5. Solve $(D^2+1)y = \operatorname{cosec} x$

① solve $(D^2+4)y = x \sin x$

Given that

$$(D^2+4)y = x \sin x \quad \text{--- (1)}$$

$$f(x)y = Q(x)$$

where $f(D) = D^2+4$ $Q(x) = x \sin x$

As $f(m) = 0$

$$m^2+4=0$$

$$m = \pm 2i \quad (\alpha=0, \beta=2)$$

Roots are complex & distinct

$$\therefore y_c = C_1 \cos 2x + C_2 \sin 2x \quad \text{--- (2)}$$

To find P.I. =

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} x \sin x \quad (m=1, a=1)$$

$$= \left[x - \frac{f'(D)}{f(D)} \right] \frac{1}{f(D)} V(x)$$

$$\left[x - \frac{2D}{D^2+4} \right] \frac{1}{D^2+4} \sin x$$

$$\text{put } D^2 = -1^2 = -1$$

$$y_p = \left[x - \frac{2D}{-1+4} \right] \frac{1}{-1+4} \sin x$$

$$= x - \frac{2}{3}$$

$$= \frac{1}{3} \left[x \sin x - \frac{2}{3} D \sin x \right]$$

$$= \frac{1}{3} x \sin x - \frac{2}{9} \cos x$$

G.S.

$$y = y_c + y_p$$

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{3} x \sin x - \frac{2}{9} \cos x$$

② Solve $(D^2 - 2D + 1)y = x e^x \sin x$.

Given that

$$(D^2 - 2D + 1)y = x e^x \sin x$$

$$f(x)y = Q(x)$$

where $f(D) = D^2 - 2D + 1$, $Q(x) = x e^x \sin x$.

$A \in \mathbb{R}$ s.t. $f(m) = 0$.

$$m^2 - 2m + 1 = 0$$

$$m^2 - m - m + 1 = 0$$

$$m(m-1) - 1(m-1) = 0$$

$$(m-1)(m-1)$$

$$m = 1, 1$$

roots are real and equal.

$$y_c = e^x (C_1 + C_2 x)$$

To find P.I :-

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} x e^x \sin x$$

$$= \frac{1}{f(D)} e^x (x \sin x) \quad (m=1, a=1)$$

$$= e^x \frac{1}{f(D+1)} (x \sin x)$$

$$= e^x \frac{1}{(D+1-1)^2} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x$$

$$= e^x \left[x - \frac{2D}{D^2} \right] \frac{1}{D^2} \sin x$$

$$= e^x \left[x - \frac{2D}{-1} \right] \frac{1}{-1} \sin x$$

$$= e^x [x + 2D] -1 \sin x$$

$$= e^x [-x \sin x - 2D \sin x]$$

$$= e^x [-x \sin x - 2 \cos x]$$

Q. 5.

$$y = y_E + y_P$$

$$y = (c_1 + c_2 x)e^x + e^x [-x \sin x - 2 \cos x]$$
$$= (c_1 + c_2 x)e^x + e^x [x \sin x + 2 \cos x]$$

⑧ Solve $(D^2 - 1)y = x^2 \cos x$.

Given that

$$(D^2 - 1)y = x^2 \cos x.$$

$$f(x)y = Q(x).$$

$$f(m) = 0$$

$$D^2 - 1 = 0$$

$$D^2 = 1$$

$$D = \pm 1$$

$$+1, -1$$

$$y = c_1 e^x + c_2 e^{-x}$$

roots are real and distinct

To find P.I

$$y_P = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2 - 1} [x^2 \cos x] \quad (m=2, a=1)$$

$$= \frac{1}{D^2 - 1} [x^2 (\text{R.P. of } e^{ix})]$$

$$= \text{R.P. of } \left[\frac{1}{D^2 - 1} e^{ix} x^2 \right]$$

$$= \text{R.P. of } \left[e^{ix} \frac{1}{(D+i)^2 - 1} x^2 \right]$$

$$= \text{R.P. of } \left[e^{ix} \frac{1}{D^2 - 1 + 2iD - 1} x^2 \right]$$

$$= \text{R.P. of } \left[e^{ix} \frac{1}{D^2 + 2iD - 2} x^2 \right]$$

$$= \text{R.P. of } \left[\frac{e^{ix}}{-2} \left[1 - \left(\frac{D^2 + 2iD}{2} \right) \right] x^2 \right]$$

$$= \text{R.P. of } \left[\frac{-e^{ix}}{2} \left[1 - \left(\frac{D^2 + 2iD}{2} \right) \right] x^2 \right]$$

$$= \text{R.P. of } \left[\frac{-e^{ix}}{2} \left[1 + \frac{D^2 + 2iD}{2} + \left(\frac{D^2 + 2iD}{2} \right)^2 \right] x^2 \right]$$

$$= \text{R.P. of } \left[\frac{-e^{ix}}{2} \left[1 + \frac{D^2 + 2iD}{2} + \left(\frac{4(-1)D^2}{4} \right) \right] x^2 \right]$$

$$= \text{R.P. of } \left[\frac{-e^{ix}}{2} \left[1 + \frac{D^2}{2} + iD - D^2 \right] x^2 \right]$$

$$= \text{R.P. of } \left[\frac{-e^{ix}}{2} \left[x^2 + \frac{1}{2}(2) + i(2x) - 2 \right] \right]$$

$$= \frac{-1}{2} \text{R.P. of } \left[(\cos x + i \sin x) ((x^2 - 1) + i(2x)) \right]$$

$$= \frac{-1}{2} \text{R.P. of } \left[(x^2 - 1) \cos 2x - 2x + i(2x \cos x + (x^2 - 1) \sin 2x) \right]$$

$$= \frac{-1}{2} \left[(x^2 - 1) \cos 2x - 2x \right]$$

$$= \frac{(1 - x^2)}{2} \cos 2x + x$$

Q. 5.

$$y = y_c + y_p$$

$$y = C_1 e^x + C_2 e^{-x} + \frac{(1-x^2)}{2} \cos x + x.$$

④. solve $(D^2+4)y = 4 \tan 2x$

Q. 7

$$(D^2+4)y = 4 \tan 2x$$

$$f(D) = D^2+4 \quad Q(x) = 4 \tan 2x$$

A. e

$$f(m) = 0$$

$$D^2+4 = 0$$

$$D^2 = -4$$

$$D = \pm \sqrt{-4}$$

$$D = \pm 2i$$

$$D = \pm 2i$$

General method to find P.I. :-

$$\text{If } f(D) = (D-\alpha_1)(D-\alpha_2)(D-\alpha_3) \dots (D-\alpha_n)$$

Then P.I.

$$y_p = \frac{1}{f(D)} Q(x)$$

$$y_p = \frac{Q(x)}{(D-\alpha_1)(D-\alpha_2) \dots (D-\alpha_n)}$$

$$= \left[\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \frac{A_3}{D-\alpha_3} + \dots + \frac{A_n}{D-\alpha_n} \right] Q(x)$$

$$= A_1 \frac{1}{D-1} Q(x) + A_2 \frac{1}{D-3} Q(x) + \dots + A_n \frac{1}{D-n} Q(x) \\
+ A_1 e^{d_1 x} \int Q(x) e^{-d_1 x} dx + A_2 e^{d_2 x} \int Q(x) e^{-d_2 x} dx + \dots + A_n e^{d_n x} \int Q(x) e^{-d_n x} dx.$$

① Solve $(D^2 + 4D + 3)y = e^{e^x}$

$$f(D)y = Q(x)$$

$$f(D) = D^2 + 4D + 3 \quad Q(x) = e^{e^x}$$

A.E

$$f(m) = 0$$

$$m^2 + 4m + 3 = 0$$

$$m^2 + 3m + m + 3 = 0$$

$$m(m+3) + 1(m+3) = 0$$

$$(m+1)(m+3) = 0$$

$$m = -1, -3$$

Roots are real and distinct.

$$y_c = C_1 e^{-x} + C_2 e^{-3x} \quad \text{--- (2)}$$

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{(D-1)(D-3)} Q(x)$$

$$= \frac{1}{2} \left[\frac{1}{(D-3)} - \frac{1}{(D-1)} \right] Q(x)$$

$$= \frac{1}{2} \left[\frac{1}{D-3} e^{e^x} - \frac{1}{D-1} e^{e^x} \right]$$

$$y_p = \frac{1}{D^2 + 4D + 3} e^{ex}$$

$$= \frac{1}{(D+1)(D+3)} e^{ex}$$

$$= \frac{1}{2} \left[\frac{1}{D+1} - \frac{1}{D+3} \right] e^{ex}$$

$$= \frac{1}{2} \left[\frac{1}{D+1} e^{ex} - \frac{1}{D+3} e^{ex} \right]$$

$$= \frac{1}{2} [y_{p1} - y_{p2}] \quad \text{--- (3)}$$

Consider

$$y_{p1} = \frac{1}{D+1} e^{ex}$$

$$= e^{-x} \int e^{ex} e^x dx$$

$$= e^{-x} \int e^t dt \quad (\text{put } e^x = t)$$

$$= e^{-x} e^t$$

$$= e^{-x} e^{ex} \quad \text{--- (a)}$$

$$y_{p2} = \frac{1}{D+3} e^{ex}$$

$$= e^{-3x} \int e^{ex} e^{3x} dx$$

$$= e^{-3x} \int e^{ex} (e^x)^2 e^x dx$$

$$= e^{-3x} \int e^t t^2 dt \quad (\text{put } e^x = t)$$

$$= e^{-3x} \int t^2 e^t dt$$

$$= e^{3x} \int 4^x e^x = 4^x e^x + 4^x e^x$$

$$= e^{3x} e^x [4^x - 2e^x + 2]$$

$$= e^{3x} e^x [e^{4x} - 2e^{3x} + 2]$$

$$y_p = e^{e^x} [e^{-x} - 2e^{-3x} + 2e^{-3x}] \quad \text{--- (6)}$$

Sub 3a and 3b

$$= \frac{1}{2} \left[\cancel{e^{-x}/e^x} - \cancel{e^x/e^x} + 2e^x e^{-3x} + 2e^x e^{-3x} \right]$$

$$= \frac{1}{2} 2e^{0x} [e^{-2x} - e^{-3x}]$$

$$= e^{e^x} [e^{-2x} - e^{-3x}] \quad \text{--- (7)}$$

General solution.

$$y = y_c + y_p$$

$$= C_1 e^{-x} + C_2 e^{-3x} + e^{e^x} [e^{-2x} - e^{-3x}]$$

② $(D^2 - 3D + 2)y = \sin(e^{-x})$

Given that

$$f(x) = D^2 - 3D + 2 \quad g(x) = \sin(e^{-x})$$

A.E

$$f(m) = 0$$

$$D^2 - 3D + 2 = 0$$

$$D^2 - 2D - D + 2 = 0$$

$$(D(D-2) - 1(D-2) = 0$$

$$(D-1)(D-2)$$

$$D = 1, 2.$$

Roots are distinct and real

$$y_c = c_1 e^x + c_2 e^{2x} \quad \text{--- (2)}$$

To find P.I. :-

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2 - 3D + 2} \sin e^{-x}$$

$$= \frac{1}{(D-1)(D-2)} \sin e^{-x}$$

$$= \left[\frac{1}{(D-2)} - \frac{1}{(D-1)} \right] \sin e^{-x}$$

$$= \left[\frac{1}{D-2} \sin e^{-x} - \frac{1}{D-1} \sin e^{-x} \right]$$

$$y_p = [y_{p1} - y_{p2}] \quad \text{--- (3)}$$

Consider

$$y_{p1} = \frac{1}{D-2} \sin e^{-x}$$

$$= e^{2x} \int \sin e^{-x} e^{-x} dx$$

$$= e^{2x} \int \sin(e^{-x}) e^{-x} e^{-x} dx$$

$$\left(\begin{array}{l} \text{put } e^{-x} = t \\ -e^{-x} dx = dt \end{array} \right)$$

$$= e^{2x} \int \sin t (dt)$$

$$= -e^{2x} \int -t \sin t \, dt$$

$$= -e^{2x} [-1 (-\cos t) - 1 (-\sin t)]$$

$$= e^{2x} [\bar{e}^x \cos \bar{e}^x - \sin \bar{e}^x]$$

$$= e^x \cos \bar{e}^x - e^{2x} \sin \bar{e}^x \quad (3a)$$

$$y_{p2} = \frac{1}{0-1} \sin \bar{e}^x$$

$$= e^x \int \sin(\bar{e}^x) \bar{e}^x \, dt$$

$$= e^x \int \sin t \, (-dt)$$

$$= -e^x \int \sin t \, dt$$

$$= e^x \cos t$$

$$y_{p2} = e^x \cos \bar{e}^x \quad (3b)$$

Sub 3a, 3b in eqn (3)

$$y_p = y_{p1} - y_{p2}$$

$$y = -e^{2x} \sin \bar{e}^x$$

G.S

$$y = y_c + y_p$$

$$y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin \bar{e}^x$$

Wronskian of functions :-

Suppose $f_1(x), f_2(x), \dots, f_n(x)$ are functions possess atleast $(n-1)$ derivatives. The wronskian of these functions is denoted by $W(f)$ or $W(f_1, f_2, \dots, f_n)$ and is defined as

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ f_1''(x) & f_2''(x) & \dots & f_n''(x) \\ f_1'''(x) & f_2'''(x) & \dots & f_n'''(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{vmatrix}$$

NOTE :- If $W(x) = 0$

Then, the given functions are called linearly dependent.

→ If $W(x) \neq 0$ then, the given functions are linearly independent.

① Show that the functions $\log x, \log x^n$ are linearly dependent.

Let $f_1(x) = \log x$, & $f_2(x) = \log x^n$

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} \log x & \log x^n \\ \frac{1}{x} & \frac{1}{x^n} \cdot nx^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} \log x & n \log x \\ \frac{1}{x} & \frac{n}{x} \end{vmatrix}$$

$$= \frac{n}{x} \log x - \frac{n}{x} \log x$$

$$= 0.$$

$W(x) = 0$ then the given function is linearly dependent

② $e^x \sin x, e^x \cos x$

③ e^x, e^{2x}

General solution of $\frac{d^2 y}{dx^2} + p \frac{dy}{dx} + q(y) = R(x)$ by the method of variation of parameters.

→ Reduce the given equation to the standard form

$$\frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = R(x)$$

→ Find the solution of $\frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = 0$

and let the C.F. = $C_1 u(x) + C_2 v(x)$

→ Take P.I. = $A(x)u(x) + B(x)v(x)$ where A & B are functions of x

→ find $W = u \frac{dv}{dx} - v \frac{du}{dx}$

→ find A and B using $A = - \int \frac{v(x)R}{W} dx$ and

$$B = \int \frac{u(x)R}{W} dx$$

→ Write the general solution of the given equation is $y = (C.F.) + P.I.$

i.e. $y = C_1 u(x) + C_2 v(x) + A(x)u(x) + B(x)v(x)$

where C_1 & C_2 are constants.

$$\textcircled{1} \quad \frac{d^2 y}{dx^2} + y = \operatorname{cosec} x$$

$$(D^2 + 1)y = \operatorname{cosec} x$$

$$f(D)y = R(x)$$

$$\text{where } f(D) = D^2 + 1 \text{ \& } R(x) = \operatorname{cosec} x$$

Consider

$$f(m) = 0$$

$$m^2 + 1 = 0$$

$$m = \pm i \quad (\alpha = 0, \beta = 1)$$

$$y_c = C_1 \cos x + C_2 \sin x \quad \textcircled{2}$$

$$= C_1 u(x) + C_2 v(x)$$

$$\text{where } u(x) = \cos x \text{ \& } v(x) = \sin x$$

Assume that

$$y_p = A(x) u(x) + B(x) v(x)$$

$$= A(x) \cos x + B(x) \sin x \quad \textcircled{3}$$

$$\text{Now } W(u, v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$= 1 \neq 0$$

Now

$$A(x) = - \int \frac{v(x) R(x)}{W} dx$$

$$= - \int \frac{\sin x \operatorname{cosec} x}{1} dx$$

$$= - \int dx$$

$$= -x \quad \textcircled{4}$$

$$B(x) = \int \frac{v(x)u'(x)}{w} dx$$

$$= \int \frac{\cos x \cos x}{1} dx$$

$$= \int \cot x \, dx$$

$$= \log(\text{Pr}x) - \text{C}$$

Sub (4) & (5) Pn (4)

$$\therefore y_p = -\cos x + (\log(\sin x)) \sin x.$$

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$$y = y_c + y_p$$

$$= C_1 \cos x + C_2 \sin x - x \cos x + (\log \sin x) \sin x,$$

$$\textcircled{2} \quad \frac{d^2 y}{dx^2} + ay = \tan ax$$

① $\frac{2}{4} a^2 \sqrt{y} = \text{correct}$

$$f(0)y = R(x)$$

where

$$f(p) = p^2 + a^2 \text{ \& } R(x) = \tan x$$

Consider

$$-1 \cdot e^{j\omega} = 0$$

$$m^2 + a^2 = 0$$

$$m = \pm a^0 \quad (\alpha=0, \beta=a)$$

$$y_c = C_1 \cos ax + C_2 \sin ax \quad (2)$$

$$= c_1 u(x) + c_2 v(x)$$

where $u(x) = \csc x$ & $v(x) = \sin x$

← Assume that

$$y_p = A(x)u(x) + B(x)v(x)$$

$$= A(x) \cos ax + B(x) \sin ax \quad \text{--- (3)}$$

$$\text{Now } W(u, v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix}$$

$$= a [\cos^2 ax + \sin^2 ax]$$

$$= a \neq 0$$

Now

$$-A(x) = - \int \frac{v(x)R(x)}{W} dx$$

$$= - \int \frac{\sin ax \tan ax}{a} dx$$

$$= -\frac{1}{a} \int \frac{\sin^2 ax}{\cos ax} dx$$

$$= -\frac{1}{a} \int \left[\frac{1 - \cos^2 ax}{\cos ax} \right] dx$$

$$= -\frac{1}{a} \int [\sec ax - \cos ax] dx$$

$$= -\frac{1}{a} \left[\frac{1}{a} \log(\sec ax + \tan ax) - \frac{\sin ax}{a} \right]$$

$$= \frac{1}{a^2} \sin ax - \frac{1}{a^2} \log(\sec ax + \tan ax) \quad \text{--- (4)}$$

$$\& B(x) = \int \frac{u(x)R(x)}{W} dx$$

$$= \int \frac{\cos ax \tan ax}{a} dx$$

$$= \frac{1}{a} \int \sin ax dx$$

$$= \frac{1}{a} \left[-\frac{\cos ax}{a} \right] = -\frac{1}{a^2} \cos ax \quad \text{--- (5)}$$

$$= \left[\frac{1}{a^2} \sin x - \frac{1}{a^2} \log (\sec x + \tan x) \right] \cos x$$

$$- \frac{1}{a^2} \cos x [\sin x]$$

$$y_p = -\frac{1}{a^2} \log (\sec x + \tan x) \cos x$$

G.S

$$y = y_c + y_p$$

$$= C_1 \cos x + C_2 \sin x - \frac{1}{a^2} \log (\sec x + \tan x) \cos x$$

$$\textcircled{8} \quad \frac{d^2 y}{dx^2} + y = x \sin x$$

$$(D+1)y = x \sin x$$

$$f(D)y = x \sin x$$

$$f(m) = m^2 + 1 = 0.$$

$$m^2 = -1$$

$$(a \pm i\beta)$$

$$(a=0, \beta=1)$$

$$y_c = C_1 \cos x + C_2 \sin x \quad \text{--- (2)}$$

$$\text{Now } y_p = -A(x) \cos x + B(x) \sin x \quad \text{--- (3)}$$

$$\text{Now } W(u, v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= [\cos^2 x - \sin^2 x]$$

$$= 1 \neq 0.$$

now

$$-A(x) = - \int \frac{v(x) \cdot u'(x)}{w} dx$$

$$= - \int \frac{\sin x (x \sin x)}{1} dx$$

$$= - \int x \sin^2 x dx$$

$$= - \int x \left[\frac{1 - \cos 2x}{2} \right] dx$$

$$= \frac{-1}{2} \left[\int x dx - \int \frac{2 \cos 2x}{2} dx \right]$$

$$= \frac{-1}{2} \left[\frac{x^2}{2} - \left[x \left(\frac{\sin 2x}{2} \right) - 1 \left(\frac{-\cos 2x}{2 \times 2} \right) \right] \right]$$

$$= \frac{-1}{2} \left[\frac{x^2}{2} - \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x \right]$$

$$= \frac{1}{2} \left[\frac{x^2}{2} + \frac{x}{2} \sin 2x - \frac{1}{4} \cos 2x \right] \leftarrow (2)$$

✓

$$B(x) = \int \frac{a(x) b(x)}{w} dx$$

$$= \int \frac{\cos x \sin x}{1} dx$$

$$= \int \frac{x \cos x \sin x}{x \cdot x} dx$$

$$= \frac{1}{2} \int x (\sin 2x \cos x) dx$$

$$= \frac{1}{2} \int \frac{x}{x} (\sin 2x) dx$$

$$= \frac{1}{2} \left[\frac{x^2}{2} \left(-\frac{\cos 2x}{2} \right) - 1 \left(-\frac{\sin 2x}{4} \right) \right]$$

$$= -\frac{x}{4} \cos 2x + \frac{1}{8} \sin 2x \quad \text{--- (5)}$$

$$y_p = \left[-\frac{x^2}{4} + \frac{x}{2} \sin 2x - \frac{1}{8} \cos 2x \right] \cos x + \left[\frac{-x}{4} \cos 2x - \frac{1}{8} \sin 2x \right] \sin x$$

$$y_p = -\frac{x^2}{4} \cos x + \frac{x}{4} \sin 2x \cos x - \frac{1}{8} \cos x \cos 2x$$

$$- \frac{x}{4} \cos 2x \sin x + \frac{1}{8} \sin 2x \sin x$$

$$= -\frac{x^2}{4} \cos x$$

$$y_p = -\frac{x^2}{4} \cos x + \frac{x}{4} \left(\sin(2x - \pi) + \frac{1}{8} \cos(2x - \pi) \right)$$

$$= -\frac{x^2}{4} \cos x + \frac{x}{4} \sin x + \frac{1}{8} \cos x \quad \text{--- (6)}$$

G.S

$$y = y_c + y_p$$

$$y = c_1 \cos x + c_2 \sin x - \frac{x^2}{4} \cos x + \frac{x}{4} \sin x + \frac{1}{8} \cos x //$$

③ solve $\frac{d^2 y}{dx^2} + y = x \sin x$

④ solve $(D^2 - 3D + 2)y = e^{4x}$

⑤ solve $\frac{d^2 y}{dx^2} - y = \frac{2}{1+e^x}$

⑥ solve $(D^2 - 5D + 6)y = e^x$

⑦ solve $(D^2 - 6D + 9)y = x^{-2} e^{3x}$

⑧ solve $(D^2 - 6D + 9)y = x^{-2} e^{3x}$

$$f(x)y = R(x)$$

where $f(D) = D^2 - 6D + 9$ & $R(x) = x^{-2} e^{3x}$

Consider $f(m) = 0$

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0$$

$$m = 3, 3$$

(roots are real & equal)

$$y_c = (c_1 + c_2 x) e^{3x}$$

$$= c_1 e^{3x} + c_2 x e^{3x} \quad \text{--- (2)}$$

$$= c_1 u(x) + c_2 v(x)$$

where

$$u(x) = e^{3x} \quad v(x) = xe^{3x}$$

Consider P.I

$$y_p = -A(x)u(x) + B(x)v(x) \\ = -A(x)e^{3x} + B(x)xe^{3x} \quad \text{--- (3)}$$

Now

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix}$$

$$= e^{6x}(1+3x) - 3xe^{6x}$$

$$= e^{6x} + 3xe^{6x} - 3xe^{6x}$$

$$= e^{6x} \neq 0$$

Now

$$A(x) = - \int \frac{v(x)R(x)}{W} dx$$

$$= - \int \frac{xe^{3x} x^{-2} e^{3x}}{e^{6x}} dx$$

$$= - \int \frac{1}{x} dx$$

$$= -\log x \quad \text{--- (4)}$$

$$\text{Now } B(x) = \int \frac{u(x)R(x)}{W} dx$$

$$= \int \frac{e^{3x} x^{-2} e^{3x}}{e^{6x}} dx$$

$$= \int \frac{1}{x^2} dx$$

$$= -\frac{1}{x} \quad \text{--- (5)}$$

sub (4) & (5) in (2)

$$y_p = (-\log x) e^{3x} + \left(-\frac{1}{2}\right) x e^{3x}$$

$$= -e^{3x} [\log x + 1] \quad \text{--- (6)}$$

G.S.

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x) e^{3x} - e^{3x} [\log x + 1],$$

$$(5) \quad \frac{d^2 y}{dx^2} - y = \frac{2}{1+e^x}$$

$$(D^2 - 1)y = \frac{2}{1+e^x}$$

$$f(D)y = R(x)$$

$$f(D) = D^2 - 1 \quad R(x) = \frac{2}{1+e^x}$$

$$f(m) = 0$$

$$(m^2 - 1) = 0$$

$$-1, 1$$

where the roots are real and distinct

$$y = c_1 e^x + c_2 e^{-x}$$

where

$$u(x) = e^x, \quad v(x) = e^{-x}$$

$$y = A(x)u(x) + B(x)v(x)$$

$$y = A(x)e^x + B(x)e^{-x}$$

$$w = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^x \end{vmatrix}$$

$$= e^x [e^{-x}] - e^x [e^{-x}]$$

$$= e^0 - e^0$$

$$= 1 - 1$$

$$= 2 \neq 0$$

$$A(x) = - \int \frac{u(x) P(x)}{w}$$

$$A(x) = - \int \frac{e^{-x} \cdot \frac{2}{1+e^x}}{2}$$

$$= -\frac{1}{2} \int \frac{2}{e^{-x}(1+e^x)} dx$$

$$= - \int \frac{1}{e^x(1+e^x)} dx$$

$$= - \int \frac{1}{t(1+t)} \frac{dt}{t}$$

$$= - \int \frac{1}{t^2(1+t)} dt \rightarrow \int \left[\frac{1}{1+t} + \frac{1}{t^2} - \frac{1}{t} \right] dt$$

$$\log(1+t) - \frac{1}{t} - \log t$$

$$\text{Now } B(x) = \int \frac{u(x) P(x)}{w} dx$$

$$\log\left(\frac{1+t}{t}\right) - \frac{1}{t}$$

$$= \int \frac{e^x \cdot \frac{2}{1+e^x}}{2} dx$$

$$= \log\left(\frac{1+e^x}{e^x}\right) - \frac{1}{e^x}$$

$$= \frac{1}{2} \int \frac{2e^x}{1+e^x} dx$$

Sub ④ & ⑤ in ③

$$y_p = (-\log x) e^{3x} + \left(-\frac{1}{2}\right) x e^{3x}$$

$$= -e^{3x} [\log x + 1] - \text{①}$$

G.S.

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x) e^{3x} - e^{3x} [\log x + 1],$$

$$\textcircled{5} \quad \frac{d^2 y}{dx^2} - y = \frac{2}{1+e^x}$$

$$(D^2 - 1)y = \frac{2}{1+e^x}$$

$$-f(D)y = R(x)$$

$$-f(D) = D^2 - 1 \quad R(x) = \frac{2}{1+e^x}$$

$$-f(m) = 0$$

$$(m^2 - 1) = 0$$

$$-1, 1$$

where the roots are real and distinct

$$y = c_1 e^x + c_2 e^{-x}$$

where

$$u(x) = e^x, v(x) = e^{-x}$$

$$y = -A(x)u(x) + B(x)v(x)$$

$$y = -A(x)e^x + B(x)e^{-x}$$

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^x \end{vmatrix}$$

$$= e^x [e^{-x}] - e^x [e^{-x}]$$

$$= e^{-0} - e^{-0}$$

$$= 1 - 1$$

$$= 2 \neq 0.$$

$$A(x) = - \int \frac{u(x) p(x)}{W} dx$$

$$A(x) = - \int \frac{e^{-x} \cdot \frac{2}{1+e^x}}{2} dx$$

$$= -\frac{1}{2} \int \frac{2}{e^{-x}(1+e^x)} dx$$

$$= - \int \frac{1}{e^x(1+e^x)} dx$$

$$= - \int \frac{1}{t(1+t)} \frac{dt}{t}$$

$$= - \int \frac{1}{t^2(1+t)} dt \rightarrow \int \left[\frac{1}{1+t} + \frac{1}{t^2} - \frac{1}{t} \right] dt$$

$$\text{Now } B(x) = \int \frac{u(x) p(x)}{W} dx$$

$$= \int \frac{e^x \cdot \frac{2}{1+e^x}}{2} dx$$

$$= \frac{1}{2} \int \frac{2e^x}{1+e^x} dx$$

$$dx = \frac{dt}{t}$$

$$e^x dx = dt$$

$$dx = \frac{dt}{e^x}$$

$$dx = \frac{dt}{t}$$

$$\log(1+t) - \frac{1}{t} - \log t$$

$$\log\left(\frac{1+t}{t}\right) - \frac{1}{t}$$

$$= \log\left(\frac{1+e^x}{e^x}\right) - \frac{1}{e^x}$$

$$= \int \frac{x^2}{1+x^2} dx$$

$$\log(1+x^2) - 3$$

$$\left[\left(\frac{x^3}{1+x^2} + \frac{1}{x^2} - \frac{1}{x} \right) \right] + C$$

$$y = \left[\log \left(\frac{1+x^2}{x^2} \right) - \frac{1}{x^2} \right] (e^x) + \log(1+e^x) e^x$$

$$= e^x \log \left(1 + \frac{1}{e^x} \right) - 1 + \frac{1}{e^x} \log(1+e^x)$$

General Solution

$$y = y_1 + y_2$$

$$y = c_1 e^x + c_2 e^{-x} + e^x \log \left(1 + \frac{1}{e^x} \right) - 1 + \frac{1}{e^x} \log(1+e^x)$$

Equations reducible to Bernoulli or with Constant Coefficient:

(i) Cauchy's homogeneous linear equation:

— An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} +$$

$$+ \dots + a_{n-1} x \frac{dy}{dx} + a_n y = 0(x)$$

where $y(x)$ is a function of x to be called y .
 Homogeneous linear equation

Such equation can be reduced to linear differential equation with constant coefficients by putting

$$x = e^t \text{ or } t = \log x. \text{ Then } \frac{d}{dx} = \frac{d}{dt}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}$$

$$= \frac{dy}{dt} \cdot \frac{1}{x}$$

$$\text{i.e. } x \frac{dy}{dx} = D y$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dt} \right)$$

$$= -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{d}{dx} \left(\frac{dy}{dt} \right) \frac{dt}{dx}$$

$$= -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{d^2 y}{dt^2} \frac{dt}{dx}$$

$$= \frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{d^2 y}{dt^2} \cdot \frac{1}{x}$$

$$= \frac{1}{x^2} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right)$$

$$\text{i.e. } x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

$$\text{Similarly } x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$$

After making these substitutions in eqn (1), we get a LDE with constant coefficients which can be solved as before.

① Solve $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = \log x$

or

$$x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = \log x \quad \text{--- (1)}$$

is a Cauchy's $x=0$ e

take $x = e^t$

$t = \log x$

Then $x \frac{d}{dx} = \theta y$, $x^2 \frac{d^2}{dx^2} = \theta(\theta-1)y$

where $\theta = \frac{d}{dt}$

$$(\theta(\theta-1) - \theta + 1)y = t$$

$$(\theta^2 - \theta - \theta + 1)y = t$$

$$(\theta^2 - 2\theta + 1)y = t$$

$$f(\theta)y = Q(t)$$

$$f(m) = 0$$

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$m=1, 1$ (roots are real and equal)

$$y_c = (C_1 + C_2 t) e^t \quad \text{--- (2)}$$

$$y_c = C_1 e^t + C_2 t e^t$$

To find P.P. :-

$$y_p = \frac{1}{f(\theta)} Q(t)$$

$$= \frac{1}{(\theta-1)^2} t$$

$$= (0-1)^2 t$$

$$= (1+20)t$$

$$= -1+20(1)$$

$$= -1+20 = 19$$

Q. 5

$$y = y_c + y_p$$

$$y = (c_1 + c_2 t) e^{-t} + t + 2$$

Q. 5 of ex 7.1 is obtained by putting $t = \log x$

$$y = (c_1 + c_2 \log x) e^{\log x} + \log x + 2$$

$$y = (c_1 + c_2 \log x) x + \log x + 2$$

② $x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x$

Q. T.

$$x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x \quad \text{--- (1)}$$

$$x = e^t \quad t = \log x$$

$$[0(0-1) + 4(0) + 2]y = e^{et}$$

$$[0^2 - 0 + 4 \cdot 0 + 2]y = e^{et}$$

$$[0^2 + 3 \cdot 0 + 2]y = e^{et}$$

$$[0^2 + 2 \cdot 0 + 0 + 2] = e^{et}$$

$$m^2 + 3m + m + 2 = 0$$

$$m(m+2) + 1(m+2)$$

$$(m+1)(m+2) = 0$$

$$m = -1, -2 \quad (\text{roots are real \& different})$$

$$y_c = c_1 e^{-x} + c_2 e^{2x} \quad \text{--- (2)}$$

$$u(x) = e^{-x} \quad v(x) = e^{2x}$$

$$y_p = A(t)u(t) + B(t)v(t)$$

now

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix}$$

$$= e^{-t} [-2e^{-2t}] - [-e^{-3t}]$$

$$= -2e^{-3t} + e^{-3t}$$

$$= -e^{-3t} \neq 0$$

$$-A(t) = - \int \frac{v(t) E(t)}{W}$$

$$= - \int \frac{e^{2t} e^{et}}{-e^{-3t}} dt$$

$$= \int e^t e^{et} dt$$

$$e^t = z$$

$$e^t = z dz$$

$$= \int e^z z dz$$

$$= \int e^z dz$$

$$= e^z$$

$$= e^{et}$$

$$B(t) = \int \frac{v(t) R(t)}{w} dt$$

$$= \int \frac{e^{-t} e^{at}}{-e^{-3t}} dt$$

$$= - \int e^{at} e^{at} dt$$

$$= - \int e^{at} e^{at} e^{-t} dt$$

$$= - \int e^x x dx$$

$$= -e^x (x-1)$$

$$= -e^{at} (e^{at} - 1)$$

$$y_p = e^{at} e^{-t} - e^{at} (e^{at} - 1) e^{-2t}$$

$$y = y_h + y_p$$

$$y = c_1 e^{-t} + c_2 e^{-2t} + e^{at} e^{-t} - e^{at} e^{-t} + e^{at} e^{-2t}$$

$$y = c_1 e^{-t} + c_2 e^{-2t} + e^{at} e^{-2t}$$

$$\text{put } t = \log x$$

$$y = c_1 e^{-\log x} + c_2 e^{-2 \log x} + e^{a \log x} e^{-2 \log x}$$

$$= \frac{c_1}{x} + \frac{c_2}{x^2} + e^x \frac{1}{x^2}$$

⑤ Solve $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = \log x \sin(\log x)$

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = \log x \sin(\log x) \quad \text{--- (1)}$$

① P.S. Cauchy's LDE

take $x = e^t$ Or $t = \log x$ then

$$x \frac{dy}{dx} = \theta y, \quad x^2 \frac{d^2 y}{dx^2} = \theta(\theta-1)$$

then eqⁿ (1) P.S

$$(\theta(\theta-1) + \theta + 1)y = -t \sin t$$

$$(\theta^2 + 1)y = -t \sin t \quad \text{--- (2)}$$

$$f(m) = 0$$

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm i$$

[$\because$ roots are complex and distinct]

$$y = c_1 \cos t + c_2 \sin t \quad \text{--- (3)}$$

$$= c_1 u(t) + c_2 v(t)$$

Finding P.I Using method of Variation of parameters

$$y_p = A(t) u(t) + B(t) v(t)$$

$$= A(t) (\cos t) + B(t) \sin t \quad \text{--- (4)}$$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$= \cos^2 t + \sin^2 t$$

$$= 1 \neq 0$$

$$-A(t) = - \int \frac{v(t)R(t)}{\omega} dt$$

$$-A(t) = - \int \frac{\sin t (t \sin t)}{1} dt$$

$$= - \int t \sin^2 t dt$$

$$= - \int t \left[\frac{1 - \cos 2t}{2} \right] dt$$

$$= - \int t (1 - \cos 2t) dt$$

$$= - \frac{1}{2} \left[\int t dt - \int t \cos 2t dt \right]$$

$$= - \frac{1}{2} \left[\frac{t^2}{2} - \left[t \frac{\sin 2t}{2} + \frac{\cos 2t}{4} \right] \right]$$

$$= - \frac{t^2}{4} + \frac{t \sin 2t}{4} + \frac{\cos 2t}{8} \quad \text{--- (1)}$$

$$B(t) = \int \frac{v(t)R(t)}{\omega} dt$$

$$= \int \frac{\cos t (t \sin t)}{1} dt$$

$$= \frac{1}{2} \int 2 \cos t (t \sin t) dt$$

$$= \frac{1}{2} \int t \sin 2t dt$$

$$= \frac{1}{2} \left[t \left(-\frac{\cos 2t}{2} \right) - 1 \left(-\frac{\sin 2t}{4} \right) \right]$$

$$= -\frac{t}{4} \cos 2t + \frac{\sin 2t}{8} \quad \text{--- (2)}$$

Sub (5) (6) in (4)

$$= \left[\frac{-t^2}{4} + \frac{-t \sin 2t}{4} + \frac{\cos 2t}{8} \right] \cos t + \left[\cos t + \frac{\sin 2t}{8} \right]$$

$$= -\frac{t^2}{4} \cos t + \frac{t}{4} \sin t + \frac{1}{8} \cos t$$

6.6

$$y = y_c + y_p$$

$$y = C_1 \cos t + C_2 \sin t + \frac{t^2}{4} \cos t + \frac{t}{4} \sin t + \frac{1}{8} \cos t$$

put $t = \log x$

$$y = C_1 \cos(\log x) + C_2 \sin(\log x) + \frac{(\log x)^2}{4} \cos(\log x) + \frac{\log x}{4} \sin(\log x) + \frac{1}{8} \cos(\log x)$$

(4) Solve $x^2 \frac{d^2 y}{dx^2} + \frac{dy}{dx} - 4y = x^2 + 2 \log x$

① Solve $(1+2x)^2 \frac{d^2y}{dx^2} - 6(1+2x) \frac{dy}{dx} + 16y = 8(1+2x)^2$

$$(1+2x)^2 \frac{d^2y}{dx^2} - 6(1+2x) \frac{dy}{dx} + 16y = 8(1+2x)^2 \quad \text{①}$$

① is a Legendre's equation

— take $1+2x = e^t \Rightarrow t = \log(1+2x)$

Then

$$(1+2x) \frac{dy}{dx} = 2y$$

$$(1+2x)^2 \frac{d^2y}{dx^2} = 2^2 \theta(\theta-1)y, \text{ where } \theta = \frac{d}{dt}$$

$$\Rightarrow [4\theta(\theta-1) - 6(2\theta) + 16]y = 8(e^t)^2$$

$$[4\theta^2 - 4\theta - 12\theta + 16]y = 8e^{2t}$$

$$(4\theta^2 - 16\theta + 16)y = 8e^{2t} \quad \text{②}$$

$$\Rightarrow \theta^2 - 4\theta + 4 = 0 \quad \text{③}$$

C.F. of ③

$$\Rightarrow \text{A.E. of } \theta^2 - 4\theta + 4 = 0$$

$$4m^2 - 16m + 16 = 0$$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$m = 2$, roots are real & equal

$$\therefore \text{ sol. } (C_1 + C_2 t) e^{2t} = \text{④}$$

P.I of ①

$$y_p = \frac{1}{f(t)} Q(t)$$

$$= \frac{1}{4t^2 - 16t + 16} 8e^{2t} \quad (a=2)$$

$$\therefore f(2)=0, f'(2)=0,$$

$$\therefore y_p = \frac{-t^2}{f''(2)} 8e^{2t}$$

$$= \frac{-t^2}{8} 8e^{2t}$$

$$= -t^2 e^{2t}$$

$$f(t) = 4t^2 - 16t + 16$$

$$f'(t) = 8t - 16$$

$$f''(t) = 8$$

$$f''(2) = 8$$

B.P. is

$$y = y_c + y_p$$

$$= (C_1 + C_2 t) e^{2t} + t^2 e^{2t}$$

$$\text{put } t = \log(1+x)$$

$$= (C_1 + C_2 \log(1+x)) e^{2 \log(1+x)} + [\log(1+x)]^2 e^{2 \log(1+x)}$$

$$= [C_1 + C_2 \log(1+x)] (1+x)^2 + 2 \log(1+x) (1+x)^2$$

2) Solve $(1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4 \cos \log(1+x)$

$$(1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4 \cos \log(1+x) \quad \text{--- (1)}$$

① Is a Legendre's equation.

$$\text{--- take } 1+x = e^t \quad t = \log(1+x)$$

Then

$$(1+x) \frac{dy}{dx} = \theta y$$

$$(1+x)^2 \frac{d^2 y}{dx^2} = \theta(\theta-1)y, \text{ where } \theta = \frac{d}{dt}$$

$$\Rightarrow (\theta(\theta-1) + \theta + 1)y = 4 \cos t$$

$$(\theta^2 - \theta + \theta + 1)y = 4 \cos t$$

$$(\theta^2 + 1)y = 4 \cos t$$

$$-f(\theta)y = Q(t)$$

C.F. of (2)

$$-A.E. \quad f(m) = 0$$

$$(m^2 + 1) = 0$$

$$m^2 = -1$$

$$m = \pm i \quad (\alpha \pm i\beta)$$

$$y_c = c_1 \cos t + c_2 \sin t \quad \text{--- (2)}$$

P.I. of (2)

$$y_p = \frac{1}{f(\theta)} Q(t)$$

$$= \frac{1}{(\theta^2 + 1)} 4 \cos t \quad (a=1)$$

$$\therefore f(\theta)^2 = 0$$

$$\therefore y_p = \frac{t}{f(\theta)} 4 \cos t$$

$$= \frac{t}{2} 4 \cos t$$

$$= 2t \frac{1}{2} \cos t$$

$$= 2t \sin t$$

$\therefore$ G.S is

$$y = y_c + y_p$$

$$= c_1 \cos t + c_2 \sin t + 2t \sin t$$

$$\text{put } t = \log(1+x)$$

$$\therefore y = c_1 \cos \log(1+x) + c_2 \sin \log(1+x) + 2 \log(1+x) \sin \log(1+x)$$

② Solve $(x+1)^2 \frac{d^2 y}{dx^2} - 3(x+1) \frac{dy}{dx} + 4y = x^2 + x + 1$

$$(x+1)^2 \frac{d^2 y}{dx^2} - 3(x+1) \frac{dy}{dx} + 4y = x^2 + x + 1 \quad \text{--- (1)}$$

① is a Legendre's Equation.

$$\text{take } (1+x) = e^t \Rightarrow t = \log(1+x)$$

$$\text{Then } (x+1) \frac{dy}{dx} = \theta y$$

$$(1+x)^2 \frac{d^2 y}{dx^2} = \theta(\theta-1)y, \text{ where } \theta = \frac{d}{dt}$$

$$\theta(\theta-1) + \theta + 1)y$$

$$(\theta(\theta-1) - 3\theta + 4)y = (x^2 + t)(e^t - 1)^2 + e^t$$

$$(\theta^2 - \theta - 3\theta + 4)y = (x^2 + t)e^{2t} - 2e^t + 1 + e^t$$

$$\theta^2 - 4\theta + 4 = (x^2 + t)e^{2t} - e^t + 1$$

$$-f(\theta)y = Q(t)$$

C.F of ①

$$-A \left(f(m) = 0 \right)$$

$$m^2 - 4m + 4 = 0$$

$$(m-2) = 0$$

$$m = 2, 2$$

$$y = (c_1 + c_2 t) e^{2t} \quad \text{--- (3)}$$

To find P.I. :-

$$y_p = \frac{1}{f(\theta)} Q(t)$$

$$= \frac{1}{f(\theta)} (e^{2t} - e^t + 1)$$

$$= \frac{1}{f(\theta)} e^{2t} - \frac{1}{f(\theta)} e^t + \frac{1}{f(\theta)} e^{0t}$$

$$= \frac{t^2}{f''(\theta)} e^{2t} - \frac{1}{f(\theta)} e^t + \frac{1}{f(\theta)} 1$$

$$y_p = \frac{t^2}{2} e^{2t} - \frac{1}{1} e^t + \frac{1}{4} \quad \text{--- (4)}$$

$$f(\theta) = \theta^2 - 4\theta + 4$$

$$f(1) = 1$$

$$f(\theta) = 4$$

$$f'(\theta) = 2\theta - 4$$

$$f''(\theta) = 2$$

$$f''(2)$$

Q. 5. of (2) is

$$y = y_c + y_p$$

$$= (c_1 + c_2 t) e^{2t} + \frac{t^2}{2} e^{2t} - e^t + \frac{1}{4}$$

put $t = \log(1+x)$

$$y = (c_1 + c_2 \log(1+x)) e^{2 \log(1+x)} + \frac{[\log(1+x)]^2}{2}$$

$$e^{2 \log(1+x)} = e^{\log(1+x)^2} = (1+x)^2$$

$$= (c_1 + c_2 \log(1+x)) (1+x)^2 + \log(1+x) (1+x)^2 - (1+x) + \frac{1}{4}$$

Solve

$$(3x+2)^2 \frac{d^2 y}{dx^2} + 3(3x+2) \frac{dy}{dx} - 36y = 3x^2 + 4x + 1$$

① P.S. - a Legendre's equation

Take $(3x+2) = e^t \Rightarrow t = \log(3x+2)$

$$x = \frac{e^t - 2}{3}$$

Then

$$(3x+2) \frac{dy}{dx} = 3\theta y, \text{ where } \theta = \frac{d}{dt}$$

$$(3x+1)^2 \frac{d^2 y}{dx^2} = 3^2 \theta(\theta-1)y$$

$$\left(\frac{3}{3} \theta(\theta-1) + 3(3\theta) - 36 \right) y = 3 \left[\frac{e^t - 2}{3} \right]^2 + 4 \left[\frac{e^t - 2}{3} \right] + 1$$

$$(9\theta^2 - 9\theta + 9\theta - 36) y = \frac{3}{9} [e^{2t} - 4e^t + 4] + \frac{4}{3} [e^t - 2] + 1$$

$$(9\theta^2 - 36) y = \frac{1}{3} e^{2t} - \frac{4}{3} e^t + \frac{4}{3} + \frac{4}{3} e^t - \frac{8}{3} + 1$$

$$(9\theta^2 - 36) y = \frac{1}{3} e^{2t} + \frac{4}{3} - \frac{8}{3} + 1$$

$$(9\theta^2 - 36) y = \frac{1}{3} e^{2t} - \frac{1}{3} \quad \text{--- (2)}$$

It is in the form of $f(\theta)y = Q(t)$

To find C.F

Consider A.E. P.S.

$$f(m) = 0$$

$$9m^2 - 36 = 0$$

$$9(m^2 - 4) = 0$$

$$m^2 - 4 = 0$$

$$m = 2, 2$$

$$y_c = \frac{(c_1 - 1)e^{2t} + c_2 e^{-2t}}{c_1 e^{2t} - c_2 e^{-2t}} \quad \text{--- (3)}$$

To find P.T.

$$y_p = \frac{1}{f(0)} Q(t)$$

$$= \frac{1}{f(0)} \left[\frac{1}{3} e^{2t} - \frac{1}{3} \right]$$

$$= \frac{1}{90^2 - 36} \left[\frac{1}{3} (e^{2t} - 1) \right]$$

$$= \frac{1}{3} \left[\frac{1}{90^2 - 36} (e^{2t} - 1) \right]$$

$$= \frac{1}{3} \left[\frac{1}{90^2 - 36} e^{2t} - \frac{1}{90^2 - 36} e^{0t} \right]$$

$$= \frac{1}{3} \left[\frac{t^2}{f''(0)} - \frac{1}{f(0)} \right]$$

$$= \frac{1}{3} \left[\frac{t^2}{18} - \frac{1}{-36} \right]$$

$$= \frac{1}{3} \left[\frac{t^2}{18} + \frac{1}{36} \right]$$

$$= \frac{1}{54} \left[t^2 + \frac{1}{2} \right]$$

$$f(0) = 90^2 - 36$$

$$f(0) = -36$$

$$f'(0) = 180$$

$$f''(0) = 18$$

$$f''(2) = 18 \cdot 36$$

$$= \frac{1}{108} [2t^2 + 1] \quad \text{--- (4)}$$

Q. 6.

$$= \frac{1}{3} \left[\frac{t}{36} - \frac{1}{-36} \right]$$

$$= \frac{1}{108} [t + 1]$$

Q. 6.

$$y = y_c + y_p$$

$$= C_1 e^{-2t} + C_2 e^{2t} + \frac{1}{108} [t + 1] \quad \text{--- (5)}$$

take $t = \log(3x+2)$ in (5)

$$= C_1 e^{-2 \log(3x+2)} + C_2 e^{2 \log(3x+2)} + \frac{1}{108} (\log(3x+2) + 1)$$

$$= \frac{C_1}{(3x+2)^2} + (3x+2)^2 + \frac{1}{108} (\log(3x+2) + 1)$$

UNIT - III

LAPLACE TRANSFORMS

- 1) L.T of standard functions
- 2) L.T theorem
- 3) I.L.T
- 4) convolution theorem
- 5) Solving ODE by L.T method.

Integral transform:

An improper integral $\int_{-\infty}^{\infty} K(s, t) f(t) dt$ is called the improper integral if it is convergent where

$K(s, t)$ is a known function of s & t and is called the kernel of a transform. If we take and is denoted by $\bar{f}(s)$

$$\text{i.e., } \int_{-\infty}^{\infty} K(s, t) f(t) dt = \bar{f}(s) \quad \text{--- ①}$$

$$\text{If we take } K(s, t) = \begin{cases} e^{-st}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$\therefore$ ① can be written as

$$\bar{f}(s) = \int_{-\infty}^0 K(s, t) f(t) dt + \int_0^{\infty} K(s, t) f(t) dt$$

$$= \int_{-\infty}^0 0 f(t) dt + \int_0^{\infty} e^{-st} f(t) dt$$

$$\Rightarrow \bar{f}(s) = \int_0^{\infty} e^{-st} f(t) dt \text{ is called the L.T}$$

Laplace Transform:

Suppose $f(t)$ is a function of positive values of t then laplace transform of $f(t)$ is denoted by $L\{f(t)\}$ and it is defined as

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = \bar{f}(s)$$

$$L\{f(t)\} = \bar{f}(s)$$

$\Rightarrow f(t) = L^{-1}\{\bar{f}(s)\}$ is called I.L.T

The symbol L is a Laplace transform operator which transforms $f(t)$ into $\bar{f}(s)$. and the symbol L^{-1} is called the Inverse Laplace transform operator which transform $\bar{f}(s)$ into $f(t)$.

Existence of L.T :

Laplace transform of a function $f(t)$ exist if the following sufficient conditions are satisfy

1) $f(t)$ is piece wise continuous or sectionally continuous in an interval $a \leq t \leq b$

2) $f(t)$ is of exponential order i.e.,

If $\lim_{t \rightarrow \infty} e^{-at} f(t) < \infty$ (finite) then $f(t)$ is

said to be of exponential order a .

Linear property:

$$\text{If } L\{f(t)\} = \bar{f}(s) \text{ \& } L\{g(t)\} = \bar{g}(s)$$

then

$$L\{af(t) + bg(t)\} = a\bar{f}(s) + b\bar{g}(s)$$

where

a & b are arbitrary constant.

Given that

$$L\{f(t)\} = \bar{f}(s) \text{ \& } L\{g(t)\} = \bar{g}(s)$$

$$\Rightarrow L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$\text{ \& } L\{g(t)\} = \int_0^{\infty} e^{-st} g(t) dt$$

Consider

$$L\{af(t) + bg(t)\} = \int_0^{\infty} e^{-st} [af(t) + bg(t)] dt$$

$$= \int_0^{\infty} e^{-st} (af(t)) dt + \int_0^{\infty} e^{-st} (bg(t)) dt$$

$$= a \int_0^{\infty} e^{-st} f(t) dt + b \int_0^{\infty} e^{-st} g(t) dt$$

$$= aL\{f(t)\} + bL\{g(t)\}$$

$$= a\bar{f}(s) + b\bar{g}(s)$$

Laplace transforms of standard functions:

1) $L\{K\} = \frac{K}{s}$, $s > 0$, K is a constant

$$\therefore L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$L\{K\} = \int_0^{\infty} e^{-st} K dt$$

$$= K \int_0^{\infty} e^{-st} dt$$

$$= K \left[\frac{e^{-st}}{-s} \right]_0^{\infty}$$

$$= -\frac{K}{s} [e^{-\infty} - e^0] \quad \left[\begin{array}{l} \because e^{-\infty} = 0 \\ e^0 = 1 \end{array} \right]$$

$$= -\frac{K}{s} [0 - 1]$$

$$= \frac{K}{s}$$

$$\therefore \boxed{L\{K\} = \frac{K}{s}}$$

2) $L\{e^{at}\} = \frac{1}{s-a}$, $s-a > 0$ or $s > a$

$$\therefore L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$L\{e^{at}\} = \int_0^{\infty} e^{-st} e^{at} dt$$

$$= \int_0^{\infty} e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty}$$

$$= \frac{-1}{s-a} [e^{-\infty} - e^0]$$

3)

$$= \frac{-1}{s-a} [0-1]$$

$$= \frac{1}{s-a}, s > a$$

$$\therefore \boxed{L\{e^{at}\} = \frac{1}{s-a}}$$

$$3) L\{e^{-at}\} = \frac{1}{s+a}, s+a > 0 \text{ or } s > -a$$

$$\therefore L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$L\{e^{-at}\} = \int_0^{\infty} e^{-st} e^{-at} dt$$

$$= \int_0^{\infty} e^{-(s+a)t} dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty}$$

$$= \frac{-1}{s+a} [e^{-\infty} - e^0]$$

$$= \frac{-1}{s+a} [0-1]$$

$$= \frac{1}{s+a}, s > -a$$

$$\therefore \boxed{L\{e^{-at}\} = \frac{1}{s+a}}$$

$$4) L\{\sin at\} = \frac{a}{s^2 + a^2}, s > 0$$

$$\therefore \sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$L\{\sin at\} = L\left\{\frac{e^{iat} - e^{-iat}}{2i}\right\}$$

$$= \frac{1}{2i} [L\{e^{iat}\} - L\{e^{-iat}\}]$$

$$= \frac{1}{2i} \left[\frac{1}{s - ai} - \frac{1}{s + ai} \right]$$

$$= \frac{1}{2i} \left[\frac{s + ai - (s - ai)}{(s - ai)(s + ai)} \right]$$

$$= \frac{1}{2i} \left[\frac{s + ai - s + ai}{s^2 + a^2} \right]$$

$$= \frac{2ai}{2i(s^2 + a^2)}$$

$$= \frac{a}{s^2 + a^2}, s > 0$$

$$\therefore \boxed{L\{\sin at\} = \frac{a}{s^2 + a^2}}$$

$$L\{\sin at\} = \int_0^{\infty} e^{-st} \sin at \, dt \quad \left[\begin{array}{l} \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{e^{-ax}}{a^2 + b^2} (a \sin bx - b \cos bx) \\ \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{e^{-ax}}{a^2 + b^2} (a \cos bx + b \sin bx) \end{array} \right]$$

$$= \left[\frac{e^{-st}}{s^2 + a^2} [-s \sin at - a \cos at] \right]_0^{\infty}$$

$$= 0 - \frac{1}{s^2 + a^2} (-s(0) - a(1))$$

$$= \frac{a}{s^2 + a^2} //$$

$$5) L\{\sinh at\} = \frac{a}{s^2 - a^2}, s > |a|$$

$$\therefore \sinh at = \frac{e^{at} - e^{-at}}{2}$$

$$L\{\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= \frac{1}{2} [L\{e^{at}\} - L\{e^{-at}\}]$$

$$= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[\frac{s+a - (s-a)}{(s-a)(s+a)} \right]$$

$$= \frac{1}{2} \left[\frac{s+a - s+a}{s^2 - a^2} \right]$$

$$= \frac{1}{2} \left[\frac{2a}{s^2 - a^2} \right]$$

$$= \frac{a}{s^2 - a^2}$$

$$\therefore L\{\sinh at\} = \frac{a}{s^2 - a^2}$$

$$\left[\begin{aligned} \sin ax &= \frac{e^{iax} - e^{-iax}}{2i} \\ \sinh ax &= \frac{e^{ax} - e^{-ax}}{2} \\ \cos ax &= \frac{e^{iax} + e^{-iax}}{2} \\ \cosh ax &= \frac{e^{ax} + e^{-ax}}{2} \end{aligned} \right]$$

$$6) L\{\cos at\} = \frac{s}{s^2 + a^2}, s > 0$$

$$\therefore \cos at = \frac{e^{iat} + e^{-iat}}{2}$$

$$\Rightarrow L\{\cos at\} = L\left\{\frac{e^{iat} + e^{-iat}}{2}\right\}$$

$$= \frac{1}{2} [L\{e^{iat}\} + L\{e^{-iat}\}]$$

$$= \frac{1}{2} \left[\frac{1}{s-ai} + \frac{1}{s+ai} \right]$$

$$= \frac{1}{2} \left[\frac{s+ai + s-ai}{s^2+a^2} \right]$$

$$= \frac{\cancel{2}s}{\cancel{2}(s^2+a^2)} = \frac{s}{s^2+a^2}$$

$$\therefore \boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$$

(or)

$$L\{\cos at\} = \int_0^{\infty} e^{-st} \cos at \, dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2} [-s \cos at + a \sin at] \right]_0^{\infty}$$

$$= 0 - \frac{1}{s^2+a^2} (-s(1) + a(0))$$

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$7) L\{\cosh at\} = \frac{s}{s^2-a^2}, \quad s > |a|$$

$$\therefore \cosh at = \frac{e^{at} + e^{-at}}{2}$$

$$\Rightarrow L\{\cosh at\} = L\left\{ \frac{e^{at} + e^{-at}}{2} \right\}$$

$$= \frac{1}{2} [L\{e^{at}\} + L\{e^{-at}\}]$$

$$= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right]$$

$$= \frac{1}{2} \left[\frac{s+a + s-a}{s^2-a^2} \right]$$

$$= \frac{KS}{s(s^2 - a^2)}$$

$$= \frac{s}{s^2 - a^2}$$

$$\boxed{L\{\cosh at\} = \frac{s}{s^2 - a^2}}$$

$$8) L\{t\} = \frac{1}{s^2}, s > 0$$

$$\therefore L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$L\{t\} = \int_0^{\infty} e^{-st} t dt$$

$$= \int_0^{\infty} t e^{-st} dt$$

$$= \left[t \left(\frac{e^{-st}}{-s} \right) - 1 \left(\frac{e^{-st}}{s^2} \right) \right]_0^{\infty}$$

$$= 0 - \left[0 - \frac{1}{s^2} \right] = \frac{1}{s^2}$$

$$\therefore \boxed{L\{t\} = \frac{1}{s^2}, s > 0}$$

$$9) L\{t^n\} = \frac{n!}{s^{n+1}}, n > 0, s > 0$$

$$= \frac{n!}{s^{n+1}}, \text{ if } ne^{st}, s > 0$$

$$\therefore L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$L\{t^n\} = \int_0^{\infty} e^{-st} t^n dt$$

$$= \int_0^{\infty} e^{-st} t^{(n+1)-1} dt$$

$$\begin{aligned} & \lim_{t \rightarrow \infty} e^{-st} f(t) \\ &= \lim_{t \rightarrow \infty} e^{-st} t \\ &= \lim_{t \rightarrow \infty} \frac{t}{e^{st}} = \frac{\infty}{\infty} \\ &= \lim_{t \rightarrow \infty} \frac{1}{s e^{st}} = \frac{1}{e^{\infty}} = \frac{1}{\infty} \\ &= 0 \end{aligned}$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\Rightarrow \text{If } n > 0 \\ \Rightarrow L\{t^n\} = \frac{n\Gamma(n)}{s^{n+1}}$$

$$\text{If } n \in \mathbb{Z}^+ \\ \Rightarrow L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt \\ \int_0^\infty e^{-kt} t^{n-1} dt = \frac{\Gamma(n)}{k^n}$$

$$\text{If } n > 0, \Gamma(n+1) = n\Gamma(n)$$

$$\text{If } n \in \mathbb{Z}^+, \Gamma(n+1) = n!$$

Standard forms of Laplace transforms.

$$1) L\{k\} = \frac{k}{s}$$

$$2) L\{e^{at}\} = \frac{1}{s-a}$$

$$3) L\{e^{-at}\} = \frac{1}{s+a}$$

$$4) L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$5) L\{\sinh at\} = \frac{a}{s^2 - a^2}$$

$$6) L\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$7) L\{\cosh at\} = \frac{s}{s^2 - a^2}$$

$$8) L\{t\} = \frac{1}{s^2}$$

$$9) L\{t^n\} = \frac{n\Gamma(n)}{s^{n+1}}, \quad n > 0, s > 0$$

$$= \frac{n!}{s^{n+1}}, \quad n \in \mathbb{Z}^+, s > 0$$

Find the Laplace transforms of following functions.

1) $e^{3t} - 2e^{-3t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9$.

Given that

$$f(t) = e^{3t} - 2e^{-3t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9$$

Applying L.T on B.S

$$\Rightarrow L\{f(t)\} = L\{e^{3t}\} - 2L\{e^{-3t}\} + L\{\sin 2t\} + L\{\cos 3t\} \\ + L\{\sinh 3t\} - 2L\{\cosh 4t\} + 9L\{1\}$$

$$= \frac{1}{s-3} - 2\frac{1}{s+3} + \frac{2}{s^2+4} + \frac{s}{s^2-9} + \frac{3}{s^2-9} - 2\frac{s}{s^2-16} + \frac{9}{s}$$

2) $f(t) = \begin{cases} 0, & 0 < t < 1 \\ 1, & 1 < t < 2 \\ 2, & t \geq 2 \end{cases}$

$$\therefore L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$= \int_0^1 e^{-st} f(t) dt + \int_1^2 e^{-st} f(t) dt + \int_2^{\infty} e^{-st} f(t) dt$$

$$= 0 + \int_1^2 e^{-st} (1) dt + \int_2^{\infty} e^{-st} (2) dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_1^2 + 2 \left[\frac{e^{-st}}{-s} \right]_2^{\infty}$$

$$= \frac{-1}{s} [e^{-2s} - e^{-s}] - \frac{2}{s} [e^{-\infty} - e^{-2s}]$$

$$= \frac{-1}{s} [e^{-2s} - e^{-s} + 0 - 2e^{-2s}]$$

$$= \frac{-1}{s} [e^{-2s} - e^{-s}] = \frac{1}{s} [e^{-2s} + e^{-s}]$$

$$3) f(t) = \sin^3 2t$$

$$\therefore \sin 3t = 3\sin t - 4\sin^3 t$$

$$\Rightarrow \sin^3 t = \frac{\sin 3t - 3\sin t}{4}$$

$$\Rightarrow \sin^3 t = \frac{3\sin t - \sin 3t}{4}$$

$$\Rightarrow \sin^3 2t = \frac{3\sin 2t - \sin 6t}{4}$$

Apply L.T on B.S

$$\Rightarrow L\{\sin^3 2t\} = \frac{1}{4} [3L\{\sin 2t\} - L\{\sin 6t\}]$$

$$= \frac{1}{4} \left[3 \times \frac{2}{s^2 + 4} - \frac{6}{s^2 + 36} \right]$$

$$= \frac{6}{4} \left[\frac{s^2 + 36 - s^2 - 4}{(s^2 + 4)(s^2 + 36)} \right]$$

$$= \frac{3}{2} \times \frac{32}{(s^2 + 4)(s^2 + 36)}$$

$$= \frac{48}{(s^2 + 4)(s^2 + 36)}$$

$$4) f(t) = \left(\sqrt{t} + \frac{1}{\sqrt{t}} \right)^3$$

$$\Rightarrow f(t) = (\sqrt{t})^3 + 3(\sqrt{t})^2 \times \frac{1}{\sqrt{t}} + 3(\sqrt{t}) \left(\frac{1}{\sqrt{t}} \right)^2 + \left(\frac{1}{\sqrt{t}} \right)^3$$

$$\Rightarrow f(t) = t^{3/2} + 3t^{1/2} + \frac{3}{t^{1/2}} + \frac{1}{t^{3/2}}$$

Applying L.T on B.S

$$L\{f(t)\} = L\{t^{3/2}\} + 3L\{t^{1/2}\} + \frac{3}{L\{t^{1/2}\}} + \frac{1}{L\{t^{3/2}\}}$$

$$= \frac{3/2 \Gamma(3/2)}{s^{3/2+1}} + 3 \cdot \frac{1/2 \Gamma(1/2)}{s^{1/2+1}} + \frac{3}{1/2 \Gamma(1/2) s^{1/2+1}} + \frac{1}{3/2 \Gamma(3/2) s^{3/2+1}}$$

$$= \frac{3\sqrt{\pi}}{4 s^{5/2}}$$

$$= \frac{3\sqrt{\pi}}{4 s^{5/2}} + \frac{3\sqrt{\pi}}{2 s^{3/2}} + \frac{6 s^{3/2}}{\sqrt{\pi}} + \frac{4 s^{5/2}}{3\sqrt{\pi}}$$

5) $f(t) = \sin 5t \cos 3t$

$$\Rightarrow f(t) = \frac{1}{2} [2 \sin 5t \cos 3t]$$

$$f(t) = \frac{1}{2} [\sin 8t + \sin 2t]$$

$$\Rightarrow L\{f(t)\} = \frac{1}{2} [L\{\sin 8t\} + L\{\sin 2t\}]$$

$$= \frac{1}{2} \left[\frac{8}{s^2 + 64} + \frac{2}{s^2 + 4} \right]$$

$$= \frac{4(s^2 + 4) + s^2 + 64}{(s^2 + 64)(s^2 + 4)}$$

$$= \frac{5s^2 + 80}{(s^2 + 64)(s^2 + 4)} //$$

Note:

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$6) f(t) = \cos 3t \cos 2t$$

$$\Rightarrow f(t) = \frac{1}{2} [2 \cos 3t \cos 2t]$$

$$f(t) = \frac{1}{2} [\cos 5t + \cos t]$$

$$\Rightarrow L\{f(t)\} = \frac{1}{2} [L\{\cos 5t\} + L\{\cos t\}]$$

$$= \frac{1}{2} \left[\frac{s}{s^2+25} + \frac{s}{s^2+1} \right]$$

$$= \frac{s}{2} \left[\frac{s^2+1+s^2+25}{(s^2+25)(s^2+1)} \right]$$

$$= \frac{s}{2} \left[\frac{2s^2+26}{(s^2+25)(s^2+1)} \right]$$

7) Prove that $f(t) = t^2$ is of an exponential order 3.

To prove the given function $f(t) = t^2$ is of an exponential order 3, it is enough to prove that

$$\lim_{t \rightarrow \infty} e^{-3t} t^2 < \infty$$

Consider

$$\lim_{t \rightarrow \infty} e^{-3t} t^2 = \lim_{t \rightarrow \infty} \frac{t^2}{e^{3t}} \left(\frac{\infty}{\infty} \right)$$

By L-Rule.

$$= \lim_{t \rightarrow \infty} \frac{2t}{3e^{3t}} \left(\frac{\infty}{\infty} \right)$$

$$\begin{aligned}
 &= \frac{\frac{3}{2} \Gamma(\frac{3}{2})}{s^{\frac{3}{2}+1}} + 3 \cdot \frac{\frac{1}{2} \Gamma(\frac{1}{2})}{s^{\frac{1}{2}+1}} + \frac{3}{\frac{1}{2} \Gamma(\frac{1}{2}) s^{\frac{1}{2}+1}} + \frac{1}{\frac{3}{2} \Gamma(\frac{3}{2}) s^{\frac{3}{2}+1}} \\
 &= \frac{\frac{3}{4} \sqrt{\pi}}{s^{\frac{5}{2}}} \\
 &= \frac{3\sqrt{\pi}}{4s^{\frac{5}{2}}} + \frac{3\sqrt{\pi}}{2s^{\frac{3}{2}}} + \frac{6s^{\frac{3}{2}}}{\sqrt{\pi}} + \frac{4s^{\frac{5}{2}}}{3\sqrt{\pi}}
 \end{aligned}$$

5) $f(t) = \sin 5t \cos 3t$

$$\Rightarrow f(t) = \frac{1}{2} [2 \sin 5t \cos 3t]$$

$$f(t) = \frac{1}{2} [\sin 8t + \sin 2t]$$

$$\Rightarrow L\{f(t)\} = \frac{1}{2} [L\{\sin 8t\} + L\{\sin 2t\}]$$

$$= \frac{1}{2} \left[\frac{8}{s^2+64} + \frac{2}{s^2+4} \right]$$

$$= \frac{4(s^2+4) + s^2+64}{(s^2+64)(s^2+4)}$$

$$= \frac{5s^2+80}{(s^2+64)(s^2+4)} //$$

Note:

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$6) f(t) = \cos 3t \cos 2t$$

$$\Rightarrow f(t) = \frac{1}{2} [2 \cos 3t \cos 2t]$$

$$f(t) = \frac{1}{2} [\cos 5t + \cos t]$$

$$\Rightarrow L\{f(t)\} = \frac{1}{2} [L\{\cos 5t\} + L\{\cos t\}]$$

$$= \frac{1}{2} \left[\frac{s}{s^2+25} + \frac{s}{s^2+1} \right]$$

$$= \frac{s}{2} \left[\frac{s^2+1+s^2+25}{(s^2+25)(s^2+1)} \right]$$

$$= \frac{s}{2} \left[\frac{2s^2+26}{(s^2+25)(s^2+1)} \right]$$

7) Prove that $f(t) = t^2$ is of an exponential order 3.

To prove the given function $f(t) = t^2$ is of an exponential order 3, it is enough to prove that

$$\lim_{t \rightarrow \infty} e^{-3t} t^2 < \infty$$

Consider

$$\lim_{t \rightarrow \infty} e^{-3t} t^2 = \lim_{t \rightarrow \infty} \frac{t^2}{e^{3t}} \left(\frac{\infty}{\infty} \right)$$

By L-Rule.

$$= \lim_{t \rightarrow \infty} \frac{2t}{3e^{3t}} \left(\frac{\infty}{\infty} \right)$$

8)

By L-Rule

$$= \lim_{t \rightarrow \infty} \frac{2}{9e^{3t}}$$
$$= \frac{2}{\infty} = 0$$

$$\Rightarrow \lim_{t \rightarrow \infty} e^{-3t} t^2 = 0 < \infty$$

$\Rightarrow f(t) = t^2$ is of an exponential order '3'

8) Prove that Laplace transform of $f(t) = e^{t^2}$ does not exist.

Given function $f(t) = e^{t^2}$ is continuous everywhere

To prove Laplace transform of $f(t) = e^{t^2}$ does not exist it is enough to prove that

$$\lim_{t \rightarrow \infty} e^{-at} f(t) = \infty$$

$$\text{Consider } \lim_{t \rightarrow \infty} e^{-at} f(t) = \lim_{t \rightarrow \infty} e^{-at} e^{t^2}$$

$$= \lim_{t \rightarrow \infty} e^{t(t-a)}$$

$$= e^{\infty}$$

$$= \infty$$

i.e., $f(t) = e^{t^2}$ is not an exponential order a .

$\therefore$ Laplace transform of e^{t^2} does not exist.

$$4) \left(\sqrt{t} + \frac{1}{\sqrt{t}} \right)^3$$

$$f(t) = (\sqrt{t})^3 + 3(\sqrt{t})^2 \times \frac{1}{\sqrt{t}} + 3\sqrt{t} \left(\frac{1}{\sqrt{t}} \right)^2 + \left(\frac{1}{\sqrt{t}} \right)^3$$

$$\Rightarrow f(t) = t^{3/2} + 3t^{1/2} + 3t^{-1/2} + t^{-3/2}$$

$$L\{f(t)\} = L\{t^{3/2}\} + 3L\{t^{1/2}\} + 3L\{t^{-1/2}\} + L\{t^{-3/2}\}$$

$$= \frac{\Gamma(3/2+1)}{s^{3/2+1}} + 3 \frac{\Gamma(1/2+1)}{s^{1/2+1}} + 3 \frac{\Gamma(-1/2+1)}{s^{-1/2+1}} + \frac{\Gamma(-3/2+1)}{s^{-3/2+1}}$$

$$= \frac{\frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}}{s^{5/2}} + \frac{3 \times \frac{1}{2} \sqrt{\pi}}{s^{3/2}} + \frac{3 \sqrt{\pi}}{s^{1/2}} + \frac{\Gamma(-1/2)}{s^{-1/2}}$$

$$= \frac{3\sqrt{\pi}}{4s^{5/2}} + \frac{3\sqrt{\pi}}{2s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}} + \frac{\Gamma(-1/2+1)}{-1/2(s^{1/2})}$$

$$= \sqrt{\pi} \left[\frac{3}{4s^{5/2}} + \frac{3}{2s^{3/2}} + \frac{3}{s^{1/2}} - 2\sqrt{s} \right]$$

Pro

Laplace - Transform theorems:

First shifting Theorem:

If $L\{f(t)\} = \bar{f}(s)$ then $L\{e^{at}f(t)\} = \bar{f}(s-a)$
or

$$= \bar{f}(s) \Big|_{s \rightarrow s-a}$$

$$= L\{f(t)\} \Big|_{s \rightarrow s-a}$$

Proof: Given that $L\{f(t)\} = \bar{f}(s)$

$$\therefore L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = \bar{f}(s)$$

$$\text{Now } L\{e^{at}f(t)\} = \int_0^{\infty} e^{-st} e^{at} f(t) dt$$

$$= \int_0^{\infty} e^{-(s-a)t} f(t) dt$$

$$= \bar{f}(s-a)$$

$$\therefore L\{e^{at}f(t)\} = \bar{f}(s-a)$$

Note:

$$L\{e^{-at}f(t)\} = \bar{f}(s+a)$$

1. Find $L\{e^{3t}\sin t\}$.

$$L\{e^{3t}\sin t\} = L\{e^{at}f(t)\}$$

$$\text{Here } a=3, f(t)=\sin t$$

$$\text{Now } L\{e^{3t}\sin t\} = L\{\sin t\} \Big|_{s \rightarrow s-3} \quad (\text{By L.T of F.S.T})$$

$$= \frac{1}{s^2+1} \Big|_{s \rightarrow s-3}$$

$$= \frac{1}{(s-3)^2+1}$$

$$= \frac{1}{s^2-6s+10}$$

2. Find $L\{e^{-2t} \cos 3t\}$

$$L\{e^{-2t} \cos 3t\} = L\{e^{-at} f(t)\}$$

$$a = 2, f(t) = \cos 3t$$

$$L\{e^{-2t} \cos 3t\} = L\{\cos 3t\} \Big|_{s \rightarrow s+2}$$

$$= \frac{s}{s^2 + 9} \Big|_{s \rightarrow s+2}$$

$$= \frac{s+2}{(s+2)^2 + 9}$$

$$= \frac{s+2}{s^2 + 4s + 13}$$

$$= \frac{s+2}{s^2 + 4s + 13} //$$

3. $L\{e^{4t} t^3\}$

$$L\{e^{4t} t^3\} = L\{e^{at} f(t)\}$$

$$a = 4, f(t) = t^3$$

$$L\{e^{4t} t^3\} = L\{t^3\} \Big|_{s \rightarrow s-4}$$

$$= \frac{3!}{s^4} \Big|_{s \rightarrow s-4}$$

$$= \frac{6}{(s-4)^4}$$

$$4. \mathcal{L}\{e^{-t}\sin 3t \cos 2t\}$$

Here $a=1$, $f(t)=\sin 3t \cos 2t$

$$\mathcal{L}\{e^{-t}\sin 3t \cos 2t\} = \mathcal{L}\{\sin 3t \cos 2t\} \Big|_{s \rightarrow s+1}$$

$$= \frac{1}{2} \mathcal{L}\{2\sin 3t \cos 2t\} \Big|_{s \rightarrow s+1}$$

$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$= \frac{1}{2} \mathcal{L}\{\sin 5t + \sin t\} \Big|_{s \rightarrow s+1}$$

$$= \frac{1}{2} \left[\mathcal{L}\{\sin 5t\} + \mathcal{L}\{\sin t\} \right] \Big|_{s \rightarrow s+1}$$

$$= \frac{1}{2} \left[\frac{5}{s^2+25} + \frac{1}{s^2+1} \right] \Big|_{s \rightarrow s+1}$$

$$= \frac{1}{2} \left[\frac{5}{(s+1)^2+25} + \frac{1}{(s+1)^2+1} \right]$$

$$= \frac{1}{2} \left[\frac{5}{s^2+2s+26} + \frac{1}{s^2+2s+2} \right]$$

(or)

$$= \frac{1}{2} \left[\frac{6s^2+30}{(s^2+25)(s^2+1)} \right] \Big|_{s \rightarrow s+1}$$

$$= \frac{3(s^2+1)+15}{[(s+1)^2+25][(s+1)^2+1]} \Big|_{s \rightarrow s+1}$$

$$= \frac{3(s+1)^2+15}{[(s+1)^2+25][(s+1)^2+1]}$$

$$5.) L\{e^{-2t} \sinh 3t\} = L\{e^{-at} f(t)\}$$

$$a=2, f(t)=\sinh 3t$$

$$L\{e^{-2t} \sinh 3t\} = L\{\sinh 3t\} \Big|_{s \rightarrow s+2}$$

$$= \frac{3}{s^2-9} \Big|_{s \rightarrow s+2}$$

$$= \frac{3}{(s+2)^2-9}$$

$$= \frac{3}{s^2+4s-5}$$

$$6.) L\{e^t \cos 2t \cos t\}$$

$$L\{e^t \cos 2t \cos t\} = L\{\cos 2t \cos t\}$$

$$a=1, f(t)=\cos 2t \cos t$$

$$L\{e^t \cos 2t \cos t\} = L\{\cos 2t \cos t\} \Big|_{s \rightarrow s-1}$$

$$= \frac{1}{2} \left[L\{2 \cos 2t \cos t\} \right] \Big|_{s \rightarrow s-1}$$

$$= \frac{1}{2} \left[L\{\cos 3t + \cos t\} \right] \Big|_{s \rightarrow s-1}$$

$$= \frac{1}{2} \left[L\{\cos 3t\} + L\{\cos t\} \right] \Big|_{s \rightarrow s-1}$$

$$= \frac{1}{2} \left[\frac{s}{s^2+9} + \frac{s}{s^2+1} \right] \Big|_{s \rightarrow s-1}$$

$$= \frac{1}{2} \left[\frac{s(s^2+1) + 5(s^2+9)}{(s^2+9)(s^2+1)} \right] \Big|_{s \rightarrow s-1}$$

$$= \frac{1}{2} \left[\frac{2s^3 + 10s}{(s^2+9)(s^2+1)} \right] \Big|_{s \rightarrow s-1}$$

$$= \left[\frac{s^3 + 5s}{(s^2+9)(s^2+1)} \right] \Big|_{s \rightarrow s-1}$$

$$= \left[\frac{(s-1)^3 + 5(s-1)}{((s-1)^2+9)((s-1)^2+1)} \right]$$

UNIT-STEP-function.

(or)

Heaviside unit function:

Unit step-function is denoted by $u(t-a)$ or $H(t-a)$ and is defined as

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

Finding Laplace transform of unit-step function:

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$L\{u(t-a)\} = \int_0^{\infty} e^{-st} u(t-a) dt$$

$$= \int_0^a e^{-st} u(t-a) dt + \int_a^{\infty} e^{-st} u(t-a) dt$$

$$= 0 + \int_0^{\infty} e^{-st} (1) dt.$$

$$\begin{aligned}
&= \int_a^{\infty} e^{-st} dt \\
&= \left[\frac{e^{-st}}{-s} \right]_a^{\infty} \\
&= \frac{-1}{s} [e^{-\infty} - e^{-as}] \\
&= \frac{-1}{s} [0 - e^{-as}]
\end{aligned}$$

$$L\{u(t-a)\} = \frac{e^{-as}}{s} \quad \text{u.}$$

Second-shifting Theorem

$$\text{If } L\{f(t)\} = \bar{f}(s) \text{ \& } g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

$$\text{then } L\{g(t)\} = e^{-as} \bar{f}(s)$$

(or)

$$L\{f(t-a)H(t-a)\} = e^{-as} \bar{f}(s)$$

Proof: Given that $L\{f(t)\} = \bar{f}(s)$

$$\Rightarrow L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = \bar{f}(s)$$

$$L\{g(t)\} = \int_0^{\infty} e^{-st} g(t) dt$$

$$= \int_0^a e^{-st} g(t) dt + \int_a^{\infty} e^{-st} g(t) dt$$

$$= 0 + \int_a^{\infty} e^{-st} f(t-a) dt$$

$$\text{take } t-a = u$$

$$\Rightarrow t = u+a$$

$$\Rightarrow dt = du$$

$$\text{If } t=a, u=0$$

$$\text{If } t \rightarrow \infty, u \rightarrow \infty$$

$$\begin{aligned} L\{g(t)\} &= \int_0^{\infty} e^{-s(u+a)} f(u) du \\ &= \int_0^{\infty} e^{-su} \cdot e^{-as} f(u) du \\ &= e^{-as} \int_0^{\infty} e^{-su} f(u) du \\ &= e^{-as} \bar{f}(s) \end{aligned}$$

$$L\{f(t-a)H(t-a)\} = e^{-as} \bar{f}(s)$$

$$\text{where } H(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

$$\text{Find } L\{g(t)\} \text{ where } g(t) = \begin{cases} \cos(t - \pi/3), & t > \pi/3 \\ 0, & t < \pi/3 \end{cases}$$

$$g(t) = \begin{cases} \cos(t - \pi/3), & t > \pi/3 \\ 0, & t < \pi/3 \end{cases} = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

$$\text{Here } a = \pi/3, f(t) = \cos t$$

$$L\{f(t)\} = L\{\cos t\} = \frac{s}{s^2+1} = \bar{f}(s)$$

By L.T of S.S.T

$$L\{g(t)\} = e^{-as} \bar{f}(s)$$

$$L\{g(t)\} = e^{-\pi/3 s} \frac{s}{s^2+1}$$

Find $L\{(t-2)^3 H(t-2)\} = L\{f(t-a)H(t-a)\}$

Here $a = 2$, $f(t) = t^3$

$$L\{f(t)\} = L\{t^3\} = \frac{3!}{s^4}$$

$$= \frac{6}{s^4} = \bar{f}(s)$$

$$\therefore L\{(t-2)^3 H(t-2)\} = e^{-2s} \frac{6}{s^4} //$$

Find $L\{\sin^2(t-3)H(t-3)\} = L\{f(t-a)H(t-a)\}$

Here $a = 3$, $f(t) = \sin^2 t$

Now $L\{f(t)\} = L\{\sin^2 t\}$

$$= L\left\{\frac{1 - \cos 2t}{2}\right\}$$

$$= \frac{1}{2} [L\{1\} - L\{\cos 2t\}]$$

$$= \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2 + 4} \right]$$

$$= \frac{1}{2} \left[\frac{s^2 + 4 - s^2}{s(s^2 + 4)} \right]$$

$$= \frac{1}{2} \left[\frac{4}{s(s^2 + 4)} \right]$$

$$= \frac{2}{s(s^2 + 4)} = \bar{f}(s)$$

By L.T of s-s-T

$$L\{f(t-a)H(t-a)\} = e^{-as} \bar{f}(s)$$

$$\Rightarrow L\{\sin^2(t-3)H(t-3)\} = e^{-3s} \frac{2}{s(s^2 + 4)} //$$

1. T. of multiplication by t :

$$\text{If } \mathcal{L}\{f(t)\} = \bar{f}(s) \text{ then } \mathcal{L}\{t f(t)\} = \frac{-d}{ds} \bar{f}(s) \quad (0.3)$$

$$\mathcal{L}\{t f(t)\} = -\bar{f}'(s)$$

$$\mathcal{L}\{t f(t)\} = \frac{-d}{ds} \mathcal{L}\{f(t)\}$$

Proof: Given that $\mathcal{L}\{f(t)\} = \bar{f}(s)$

$$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\}$$

$$\Rightarrow \int_0^{\infty} e^{-st} f(t) dt = \bar{f}(s) \quad \text{--- (1)}$$

Diff (1) B.S w.r.t 's'.

$$\Rightarrow \frac{d}{ds} \int_0^{\infty} e^{-st} f(t) dt = \frac{d}{ds} \bar{f}(s).$$

$$\Rightarrow \frac{d}{ds} \int_0^{\infty} \frac{\partial}{\partial s} (e^{-st}) \cdot f(t) dt = \frac{d}{ds} \bar{f}(s).$$

$$\Rightarrow \int_0^{\infty} (e^{-st}) (-t) f(t) dt = \frac{d}{ds} \bar{f}(s)$$

$$\Rightarrow - \int_0^{\infty} e^{-st} [t f(t)] dt = \frac{d}{ds} \bar{f}(s)$$

$$\Rightarrow \mathcal{L}\{t f(t)\} = \frac{-d}{ds} \bar{f}(s) //$$

$$\therefore \boxed{\mathcal{L}\{t f(t)\} = \frac{-d}{ds} \bar{f}(s)}$$

Note:

$$\mathcal{L}\{t f(t)\} = (-1)^1 \frac{d}{ds} \bar{f}(s)$$

$$\mathcal{L}\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} \bar{f}(s)$$

$$\mathcal{L}\{t^3 f(t)\} = (-1)^3 \frac{d^3}{ds^3} \bar{f}(s)$$

In general

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \bar{f}(s)$$

$$1) L\{t \sin at\} = L\{t f(t)\}$$

$$d\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\text{Here } f(t) = \sin at$$

By L.T of multiplication by t

$$L\{t f(t)\} = -\frac{d}{ds} [L\{f(t)\}]$$

Now

$$L\{t \sin at\} = -\frac{d}{ds} L\{\sin at\}$$

$$= -\frac{d}{ds} \left[\frac{a}{s^2 + a^2} \right]$$

$$= -a \frac{d}{ds} \left[\frac{1}{s^2 + a^2} \right]$$

$$= -a \left[\frac{-1}{(s^2 + a^2)^2} (2s + 0) \right]$$

$$L\{t \sin at\} = \frac{2as}{(s^2 + a^2)^2}$$

$$2) L\{t \cos^2 t\} = L\{t f(t)\}$$

$$\text{Here } f(t) = \cos^2 t$$

By L.T of multiplication by t

$$L\{t f(t)\} = -\frac{d}{ds} L\{f(t)\}$$

$$L\{t \cos^2 t\} = -\frac{d}{ds} L\{\cos^2 t\}$$

$$= -\frac{d}{ds} L\left\{ \frac{1 + \cos 2t}{2} \right\}$$

$$= -\frac{1}{2} \left[\frac{d}{ds} \left[\frac{1}{s} + \frac{s}{s^2 + 4} \right] \right]$$

$$= \frac{-1}{2} \left[\frac{-1}{s^2} + \left(\frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2} \right) \right]$$

$$= \frac{-1}{2} \left[\frac{-1}{s^2} + \frac{4-s^2}{(s^2+4)^2} \right]$$

$$= \frac{1}{2} \left[\frac{1}{s^2} - \frac{(4-s^2)}{(s^2+4)^2} \right]$$

3) $L\{t e^{3t} \sin t\} = L\{e^{3t}(t \sin t)\} = L\{e^{at} f(t)\}$

Here $a=3$, $f(t) = t \sin t$

By L.T of F.S.T

$$L\{e^{at} f(t)\} = L\{f(t)\}_{s \rightarrow s-a}$$

$$\therefore L\{e^{3t}(t \sin t)\} = L\{t \sin t\}_{s \rightarrow s-3}$$

$$= \left[-\frac{d}{ds} L\{\sin t\} \right]_{s \rightarrow s-3}$$

$$= \left[-\frac{d}{ds} \left(\frac{1}{s^2+1} \right) \right]_{s \rightarrow s-3}$$

$$= \left[-\left(\frac{-1}{(s^2+1)^2} \cdot 2s \right) \right]_{s \rightarrow s-3}$$

$$= \left[\frac{2s}{(s^2+1)^2} \right]_{s \rightarrow s-3}$$

$$= \frac{2(s-3)}{((s-3)^2+1)^2}$$

$$= \frac{2(s-3)}{(s^2-6s+10)^2} //$$

$$4) \quad L\{t^2 \cos at\} = L\{t^2 f(t)\}$$

$$\text{Here } f(t) = \cos at$$

By L.T of multiplication by t^2

$$L\{t^2 f(t)\} = \frac{d^2}{ds^2} L\{\cos at\}$$

$$= \frac{d^2}{ds^2} \left[\frac{s}{s^2 + a^2} \right]$$

$$= \frac{d}{ds} \left[\frac{d}{ds} \left(\frac{s}{s^2 + a^2} \right) \right]$$

$$= \frac{d}{ds} \left[\frac{(s^2 + a^2)(1) - s(2s)}{(s^2 + a^2)^2} \right]$$

$$= \frac{d}{ds} \left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2} \right]$$

$$= \frac{d}{ds} \left[\frac{a^2 - s^2}{(s^2 + a^2)^2} \right]$$

$$= \frac{(s^2 + a^2)^2(-2s) - (a^2 - s^2)2(s^2 + a^2)(2s)}{(s^2 + a^2)^4}$$

$$= (s^2 + a^2) \left[\frac{-2s - (a^2 - s^2)(4s)}{(s^2 + a^2)^4} \right]$$

$$= \frac{-2s - 4sa^2 + 4s^3}{(s^2 + a^2)^3}$$

$$5) L\{t \sinh at \sin bt\}$$

$$\text{Here } f(t) = \sinh at \sin bt$$

By L.T of multiplication by t.

$$L\{t f(t)\} = -\frac{d}{ds} \{f(t)\}$$

$$\text{Now } L\{t \sinh at \sin bt\} = -\frac{d}{ds} L\{\sinh at \sin bt\}$$

$$= -\frac{d}{ds} L\left\{\left(\frac{e^{at} - e^{-at}}{2}\right) \sin bt\right\}$$

$$= -\frac{1}{2} \left[\frac{d}{ds} [L\{e^{at} \sin bt\} - L\{e^{-at} \sin bt\}] \right]$$

$$= -\frac{1}{2} \frac{d}{ds} \left[L\{\sin bt\}_{s \rightarrow s-a} - L\{\sin bt\}_{s \rightarrow s+a} \right]$$

$$= -\frac{1}{2} \frac{d}{ds} \left[\frac{3}{s^2+9} \Big|_{s \rightarrow s-2} - \frac{3}{s^2+9} \Big|_{s \rightarrow s+2} \right]$$

$$= -\frac{1}{2} \frac{d}{ds} \left[\frac{3}{(s-2)^2+9} - \frac{3}{(s+2)^2+9} \right]$$

$$= -\frac{1}{2} \frac{d}{ds} \left[\frac{3}{s^2-4s+13} - \frac{3}{s^2+4s+13} \right]$$

$$= -\frac{1}{2} \left[\frac{3(-1) \cdot (2s-4)}{(s^2-4s+13)^2} - \frac{3(-1) \cdot (2s+4)}{(s^2+4s+13)^2} \right]$$

$$= \frac{6}{2} \left[\frac{(s-2)}{(s^2-4s+13)^2} (2s-4) - \frac{(s+2)}{(s^2+4s+13)^2} (2s+4) \right]$$

$$= \frac{6}{2} \left[\frac{(s-2)}{(s^2-4s+13)^2} - \frac{(s+2)}{(s^2+4s+13)^2} \right]$$

$$= 3 \left[\frac{s-2}{(s^2-4s+13)^2} - \frac{(s+2)}{(s^2+4s+13)^2} \right]$$

$$6) \quad L\{t^2 e^{-3t} \cosh 2t\} = L\{t^2 f(t)\}$$

$$\text{Here } f(t) = e^{-3t} \cosh 2t$$

By L.T of multiplication by t^2

$$L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} L\{f(t)\}$$

$$L\{t^2 (e^{-3t} \cosh 2t)\} = \frac{d^2}{ds^2} L\{e^{-3t} \cosh 2t\}$$

$$= \frac{d^2}{ds^2} \left[L\{\cosh 2t\}_{s \rightarrow s+3} \right] \quad (\text{By L.T of F.S.T})$$

$$= \frac{d^2}{ds^2} \left[\frac{s}{s^2 - 4} \Big|_{s \rightarrow s+3} \right]$$

$$= \frac{d^2}{ds^2} \left[\frac{s+3}{(s+3)^2 - 4} \right]$$

$$= \frac{d^2}{ds^2} \left[\frac{s+3}{s^2 + 6s + 5} \right]$$

$$= \frac{d}{ds} \left[\frac{d}{ds} \left[\frac{s+3}{s^2 + 6s + 5} \right] \right]$$

$$= \frac{d}{ds} \left[\frac{(s^2 + 6s + 5)(1) - (s+3)(2s+6)}{(s^2 + 6s + 5)^2} \right] = \frac{d}{ds} \left[\frac{-s^2 - 6s - 13}{(s^2 + 6s + 5)^2} \right]$$

$$= -\frac{d}{ds} \left[\frac{s^2 + 6s + 13}{(s^2 + 6s + 5)^2} \right]$$

$$\neq \frac{d}{ds} \left[\frac{s^2 + 6s + 13}{(s^2 + 6s + 5)^2} \right]$$

$x^2 = 2x$

$$= - \left[\frac{(s^2+6s+5)^2(2s+6) - (s^2+6s+13)2(s^2+6s+5)(2s+6)}{(s^2+6s+5)^4} \right]$$

$$= - \left[\frac{(s^2+6s+5) [2(s+3)(s^2+6s+5) - 4(s^2+6s+13)(s+3)]}{(s^2+6s+5)^4} \right]$$

$$= - \frac{2(s+3) [2s^2+12s+10 - 4s^2-24s-52]}{(s^2+6s+5)^3}$$

$$= - \left[\frac{2(s+3) [-s^2-6s-21]}{(s^2+6s+5)^3} \right]$$

$$= \frac{2(s+3)(s^2+6s+21)}{(s^2+6s+5)^3}$$

$$7) L \{ t e^{2t} (\cos t - \sin t) \} = L \{ t f(t) \}$$

$$= L \{ t e^{2t} \cos t - t e^{2t} \sin t \}$$

$$= L \{ t e^{2t} \cos t \} - L \{ t e^{2t} \sin t \}$$

Laplace transform of division by t :

$$\text{If } L\{f(t)\} = \bar{f}(s) \text{ then, } L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \bar{f}(s) ds \\ = \int_s^{\infty} L\{f(t)\} ds$$

Given that $L\{f(t)\} = \bar{f}(s)$

$$\Rightarrow \bar{f}(s) = L\{f(t)\}$$

$$\Rightarrow \bar{f}(s) = \int_0^{\infty} e^{-st} f(t) dt \quad \text{--- (1)}$$

$$\int_s^{\infty} \bar{f}(s) ds = \int_s^{\infty} \left[\int_0^{\infty} e^{-st} f(t) dt \right] ds$$

Taking integration on both sides of eq (1) w.r.t s with limits as $s \rightarrow \infty$ then

we get

$$\int_s^{\infty} \bar{f}(s) ds = \int_s^{\infty} \left[\int_0^{\infty} e^{-st} f(t) dt \right] ds$$

$$= \int_0^{\infty} \left[\int_s^{\infty} e^{-st} ds \right] f(t) dt$$

$$= \int_0^{\infty} \left[\frac{e^{-st}}{-t} \right]_s^{\infty} f(t) dt$$

$$= \int_0^{\infty} \frac{1}{t} (e^{-\infty} - e^{-st}) f(t) dt$$

$$= \int_0^{\infty} \frac{1}{t} (0 - e^{-st}) f(t) dt$$

$$= \int_0^{\infty} e^{-st} \left(\frac{f(t)}{t} \right) dt = L\left\{\frac{f(t)}{t}\right\}$$

$$\Rightarrow \int_s^{\infty} \bar{f}(s) ds = L\left\{\frac{f(t)}{t}\right\}$$

$$\therefore L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \bar{f}(s) ds$$

Note:

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \bar{f}(s) ds$$

$$L\left\{\frac{f(t)}{t^2}\right\} = \int_s^{\infty} \int_s^{\infty} \bar{f}(s) ds ds$$

In general

$$L\left\{\frac{f(t)}{t^n}\right\} = \int_s^{\infty} \int_s^{\infty} \dots \int_s^{\infty} \bar{f}(s) ds \dots ds ds.$$

Ex 4

1. Find $L\left\{\frac{\sin at}{t}\right\} = L\left\{\frac{f(t)}{t}\right\}$

Here $f(t) = \sin at$

$$\text{Now } L\{f(t)\} = L\{\sin at\} = \frac{a}{s^2 + a^2} = \bar{f}(s)$$

By L.T of Division by t

$$\Rightarrow L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \bar{f}(s) ds$$

$$\Rightarrow L\left\{\frac{\sin at}{t}\right\} = \int_s^{\infty} \frac{a}{s^2 + a^2} ds$$

$$= a \int_s^{\infty} \frac{1}{s^2 + a^2} ds$$

$$= a \left[\frac{1}{a} \tan^{-1}\left(\frac{s}{a}\right) \right]_s^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \tan^{-1}\left(\tan^{\infty} \frac{\pi}{2}\right) - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$

$$2) \quad L\left\{\frac{\cos at}{t}\right\} = L\left\{\frac{f(t)}{t}\right\}$$

$$2 \sin A \cos B = \sin(A+B) - \sin(A-B)$$

Here $f(t) = \cos at$

$$\text{Now } L\{f(t)\} = L\{\cos at\} = \frac{1}{s^2 + a^2} = \bar{f}(s)$$

By L-T of division by t

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s) ds$$

$$L\left\{\frac{\cos at}{t}\right\} = \int_s^\infty \frac{1}{s^2 + a^2} ds \cdot L\left\{\frac{1}{s}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2 + a^2}\right) ds$$

$$= \frac{1}{2} \int_s^\infty \frac{2s}{s^2 + a^2} ds$$

$$= \frac{1}{2} (\log(s^2 + a^2))_s^\infty$$

$$= \frac{1}{2} (\log(\infty) - \log(s))$$

$$= \frac{1}{2} \log(s) \quad (\text{consider } \log \infty = 0 \text{ by Laplace})$$

$$= \left[\log(s) - \frac{1}{2} \log(s^2 + a^2) \right]_s^\infty$$

$$= \left[\log s - \log(s^2 + a^2)^{1/2} \right]_s^\infty$$

$$= \left[\log \left[\frac{s}{\sqrt{s^2 + a^2}} \right] \right]_s^\infty$$

$$= \log \left[\frac{s}{s \sqrt{1 + \frac{a^2}{s^2}}} \right]_s^\infty$$

$$= \log \left[\frac{1}{\sqrt{1 + \frac{a^2}{s^2}}} \right] - \log \left(\frac{1}{\sqrt{1 + \frac{a^2}{s^2}}} \right)$$

$$= \log(1) - \log\left(\frac{s}{\sqrt{s^2 + a^2}}\right) = 0 - \log\left(\frac{s}{\sqrt{s^2 + a^2}}\right) = \log\left(\frac{\sqrt{s^2 + a^2}}{s}\right)$$

$$3) \quad L\left\{\frac{\sin 3t \cos t}{t}\right\} = L\left\{\frac{f(t)}{t}\right\}$$

$$\text{Here } f(t) = \sin 3t \cos t$$

$$L\{f(t)\} = L\{\sin 3t \cos t\}$$

$$= \frac{1}{2} L\{2 \sin 3t \cos t\}$$

$$= \frac{1}{2} L\{\sin 4t + \sin 2t\}$$

$$= \frac{1}{2} [L\{\sin 4t\} + L\{\sin 2t\}]$$

$$= \frac{1}{2} \left[\frac{4}{s^2 + 16} + \frac{2}{s^2 + 4} \right] = \bar{f}(s).$$

By L.T of division by t

$$L\left\{\frac{f(t)}{t}\right\} = \frac{1}{2} \int_0^{\infty} \left[\frac{4}{s^2 + 16} + \frac{2}{s^2 + 4} \right] ds$$

$$= \frac{1}{2} \left[\int_0^{\infty} \frac{4}{s^2 + 16} ds + \int_0^{\infty} \frac{2}{s^2 + 4} ds \right]$$

$$= \frac{1}{2} \left[4 \int_0^{\infty} \frac{1}{s^2 + 4^2} ds + 2 \int_0^{\infty} \frac{1}{s^2 + 2^2} ds \right]$$

$$= \frac{1}{2} \left[4 \cdot \frac{1}{4} \tan^{-1}\left(\frac{s}{4}\right) + 2 \cdot \frac{1}{2} \tan^{-1}\left(\frac{s}{2}\right) \right]_0^{\infty}$$

$$= \frac{1}{2} \left[\tan^{-1}\left(\frac{s}{4}\right) + \tan^{-1}\left(\frac{s}{2}\right) \right]_0^{\infty}$$

$$\begin{aligned}
&= \frac{1}{2} [\tan^{-1}(\infty) - \tan^{-1}(s/4) + \tan^{-1}(\infty) - \tan^{-1}(s/2)] \\
&= \frac{1}{2} [\pi/2 - \tan^{-1}(s/4) + \pi/2 - \tan^{-1}(s/2)] \\
&= \frac{1}{2} [\cot^{-1}(s/4) + \cot^{-1}(s/2)]
\end{aligned}$$

$$4) \mathcal{L} \left\{ \frac{e^{2t} \sin t}{t} \right\} = \mathcal{L} \left\{ \frac{f(t)}{t} \right\}$$

$$\text{Here } f(t) = e^{2t} \sin t$$

$$\text{Now } \mathcal{L}\{f(t)\} = \mathcal{L}\{e^{2t} \sin t\}$$

$$= \mathcal{L}\{\sin t\}_{s \rightarrow s-2}$$

$$= \frac{1}{s^2 + 1} \Big|_{s \rightarrow s-2}$$

$$= \frac{1}{(s-2)^2 + 1}$$

$$= \frac{1}{s^2 - 4s + 5} = \bar{f}(s)$$

By L.T of division by t

$$\mathcal{L} \left\{ \frac{f(t)}{t} \right\} = \int_s^\infty \bar{f}(s) ds$$

$$\Rightarrow \mathcal{L} \left\{ \frac{e^{2t} \sin t}{t} \right\} = \int_s^\infty \frac{1}{s^2 - 4s + 5} ds$$

$$= \int_s^\infty \frac{1}{(s-2)^2 + 1} ds$$

$$= \left[\tan^{-1}(s-2) \right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s-2)$$

$$= \pi/2 - \tan^{-1}(s-2)$$

$$= \cot^{-1}(s-2)$$

$$5) \mathcal{L} \left\{ \frac{\sinh t}{t} \right\} = \mathcal{L} \left\{ \frac{f(t)}{t} \right\}$$

$$\text{Here } f(t) = \sinh t$$

$$\mathcal{L} \{f(t)\} = \mathcal{L} \{ \sinh t \}$$

$$= \frac{1}{s^2 - 1} = \bar{f}(s)$$

$$\mathcal{L} \left\{ \frac{\sinh t}{t} \right\} = \int_s^\infty \frac{1}{s^2 - 1} ds \quad \mathcal{L} \left\{ \frac{f(t)}{t} \right\} = \left[\frac{1}{s^2 - a^2} \right. \\ \left. = \int_s^\infty \frac{1}{s^2 - 1} ds \quad \left[\frac{1}{2a} \log \left(\frac{s-a}{s+a} \right) \right] \right]$$

$$= \left[\frac{1}{2} \log \left(\frac{s^2 - 1}{s+1} \right) \right]_s^\infty$$

$$= \frac{1}{2} \log \left[\frac{s(1 - \frac{1}{s})}{s(1 + \frac{1}{s})} \right]_s^\infty$$

$$= \frac{1}{2} \left[\log \left(\frac{1 - \frac{1}{\infty}}{1 + \frac{1}{\infty}} \right) - \log \left(\frac{1 - \frac{1}{s}}{1 + \frac{1}{s}} \right) \right]$$

$$= \frac{1}{2} \left[\log(1) - \log \left(\frac{s-1}{s+1} \right) \right]$$

$$= \frac{1}{2} \left[0 - \log \left(\frac{s-1}{s+1} \right) \right]$$

$$= \frac{1}{2} \left[\log \left(\frac{s+1}{s-1} \right) \right]$$

$$uv' - u'v = 1$$

Laplace transform of derivatives:

If $f(t)$ is piece wise continuous and of exponential order, then Laplace transform of 1st derivative is denoted by $L\{f'(t)\}$ and is defined as $L\{f'(t)\} = s\bar{f}(s) - f(0)$

(or)

$$= sL\{f(t)\} - f(0)$$

Proof: $\therefore L\{f(t)\} = \bar{f}(s)$

$$\Rightarrow L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$\text{Now } L\{f'(t)\} = \int_0^{\infty} e^{-st} f'(t) dt$$

$$= \left[e^{-st} \int f'(t) dt \right]_0^{\infty} - \left[\int_0^{\infty} \left[e^{-st} (-s) \cdot \int f'(t) dt \right] dt \right]$$

$$= \left[e^{-st} f(t) \right]_0^{\infty} + s \left[\int_0^{\infty} e^{-st} f(t) dt \right]$$

$$= \lim_{t \rightarrow \infty} e^{-st} f(t) - e^0 f(0) + sL\{f(t)\}$$

$$= 0 - f(0) + s\bar{f}(s)$$

$$= s\bar{f}(s) - f(0) //$$

$$\begin{array}{l} uv_1 - u'v_2 \\ + \\ \int uv = u \int v dv \\ - \int u'v \int \frac{u'}{u} \end{array}$$

Note:

$$L\{f'(t)\} = s\bar{f}(s) - f(0)$$

$$L\{f''(t)\} = s^2\bar{f}(s) - sf(0) - f'(0)$$

$$L\{f'''(t)\} = s^3\bar{f}(s) - s^2f(0) - sf'(0) - f''(0)$$

$$L\{f^{(4)}(t)\} = s^4\bar{f}(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$$

$$\vdots$$
$$L\{f^{(n)}(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \dots - f^{(n-1)}(0)$$

1. Find $L\{e^{at}\}$ using L.T of derivative.

Given that $f(t) = e^{at}$

then, $f'(t) = ae^{at}$

By L.T of first derivative

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

$$\Rightarrow L\{ae^{at}\} = sL\{e^{at}\} - e^0$$

$$\Rightarrow aL\{e^{at}\} - sL\{e^{at}\} = -1$$

$$\Rightarrow L\{e^{at}\}(a-s) = -1$$

$$\Rightarrow L\{e^{at}\} = \frac{-1}{a-s}$$

$$\Rightarrow L\{e^{at}\} = \frac{1}{s-a}$$

2. Find $L\{\cos t\}$ using L.T of derivative

Given that $f(t) = \cos t$

then $f'(t) = -\sin t$

$f''(t) = -\cos t$

$$\bar{f}(s) = L\{f(t)\}$$

By L.T of ~~first~~ second derivative

$$L\{f''(t)\} = s^2 \bar{f}(s) - sf(0) - f'(0)$$

$L\{\cos t\}$ is

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$$

$$L\{\cos t\} = s^2 L\{\cos t\} - s \cos 0^\circ + \sin 0$$

$$-L\{\cos t\} - s^2 L\{\cos t\} = -s + 0$$

$$-L\{\cos t\} (1 + s^2) = -s$$

$$L\{\cos t\} = \frac{s}{s^2 + 1}$$

3. Find $L\{t \sin t\}$ using L.T of derivative.

Given that $f(t) = t \sin t$

$$f(t) = t \sin t$$

$$f'(t) = \sin t + t \cos t$$

$$f''(t) = \cos t + \cos t + (-t \sin t)$$

$$= 2\cos t - t \sin t$$

By L.T of second derivative

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$$

$$L\{2\cos t - t \sin t\} = s^2 L\{t \sin t\} - s(0) - (0 + 0)$$

$$L\{2\cos t\} - L\{t\sin t\} - s^2 L\{t\sin t\} = 0$$

$$2L\{\cos t\} - L\{t\sin t\}(1+s^2) = 0$$

$$2 \frac{s}{s^2+1} - L\{t\sin t\}(1+s^2) = 0$$

$$L\{t\sin t\}(1+s^2) = \frac{2s}{s^2+1}$$

$$L\{t\sin t\} = \frac{2s}{(s^2+1)^2}$$

4) Find $L\{f(t)\}$ & $L\{f'(t)\}$ where $f(t) = e^{2t}t^3$

Given that $f(t) = e^{2t}t^3$

$$(i) \text{ Now } L\{f(t)\} = L\{e^{2t}t^3\}$$

$$= L\{t^3\}_{s \rightarrow s-2}$$

$$= \frac{3!}{s^4} \Big|_{s \rightarrow s-2}$$

$$= \frac{6}{(s-2)^4} = \bar{f}(s)$$

$$\therefore L\{f'(t)\} = s\bar{f}(s) - f(0)$$

$$L\{f'(t)\} = s \frac{6}{(s-2)^4} - e^0(0)$$

$$L\{f'(t)\} = \frac{6s}{(s-2)^4}$$

L.T of Integrals:

$$\text{If } L\{f(t)\} = \bar{f}(s) \text{ then } L\left\{\int_0^t f(t) dt\right\} = \frac{\bar{f}(s)}{s}$$
$$= \frac{L\{f(t)\}}{s}$$

Proof: Given that $L\{f(t)\} = \bar{f}(s)$

$$\text{take } q(t) = \int_0^t f(t) dt \quad \text{--- (1)}$$

Diff (1) w.r.t t .

$$\frac{d}{dt}(q(t)) = \frac{d}{dt} \left[\int_0^t f(t) dt \right]$$

$$q'(t) = f(t) \quad \text{--- (2)}$$

Apply L.T on both sides of (2)

$$L\{q'(t)\} = L\{f(t)\}$$

By L.T of first derivative

$$\Rightarrow s L\{q(t)\} - q(0) = \bar{f}(s)$$

$$\Rightarrow s L\left\{\int_0^t f(t) dt\right\} - \int_0^0 f(t) dt = \bar{f}(s)$$

$$\Rightarrow s L\left\{\int_0^t f(t) dt\right\} = \bar{f}(s)$$

$$\Rightarrow \boxed{L\left\{\int_0^t f(t) dt\right\} = \frac{\bar{f}(s)}{s}}$$

$$1) \text{ Find } L \left\{ \int_0^t e^{-2t} t^2 dt \right\} = L \left\{ \int_0^t f(t) dt \right\}$$

$$\text{Here } f(t) = e^{-2t} t^2$$

$$\text{Now } L \{f(t)\} = L \{e^{-2t} t^2\}$$

$$= L \{t^2\}_{s \rightarrow s+2} \quad (\text{By L.T of F.S.T})$$

$$= \frac{2}{s^3} \Big|_{s \rightarrow s+2}$$

$$= \frac{2}{(s+2)^3} = \bar{f}(s)$$

By L.T of integrals

$$L \left\{ \int_0^t f(t) dt \right\} = \frac{\bar{f}(s)}{s}$$

$$\Rightarrow L \left\{ \int_0^t e^{-2t} t^2 dt \right\} = \frac{1}{s} \left[\frac{2}{(s+2)^3} \right]$$

$$= \frac{2}{s(s+2)^3} //$$

$$2) L \left\{ \int_0^t t e^t \sin t dt \right\} = L \left\{ \int_0^t f(t) dt \right\}$$

$$\text{Here } f(t) = t e^t \sin t$$

$$\text{Now } L \{f(t)\} = L \{e^t t \sin t\}$$

$$= L \{t \sin t\}_{s \rightarrow s-1} \quad (\text{By L.T of F.S.})$$

$$= \left[-\frac{d}{ds} L \{\sin t\} \right]_{s \rightarrow s-1}$$

(By L.T of multiplication by t)

$$= \left[\frac{-d}{ds} \left[\frac{1}{s^2+1} \right] \right]_{s \rightarrow s-1}$$

$$= - \left[\frac{-1}{(s^2+1)^2} (2s) \right]_{s \rightarrow s-1}$$

$$= \left[\frac{2s}{(s^2+1)^2} \right]_{s \rightarrow s-1}$$

$$= \frac{2(s-1)}{[(s-1)^2+1]^2}$$

$$= \frac{2(s-1)}{(s^2-2s+2)}, = \bar{f}(s)$$

By L.T of integrals

$$L \left\{ \int_0^t f(t) dt \right\} = \frac{\bar{f}(s)}{s}$$

$$\Rightarrow L \left\{ \int_0^t e^t \sin t dt \right\} = \frac{2(s-1)}{s(s^2-2s+2)} //$$

Evaluation of improper integral using the definition of L.T:

1) Evaluate $\int_0^{\infty} e^{-3t} t^4 dt = \int_0^{\infty} e^{-st} f(t) dt$

$$\int_0^{\infty} e^{-st} f(t) dt = L\{f(t)\}$$

where $s=3, f(t)=t^4$

$$\therefore \int_0^{\infty} e^{-3t} t^4 dt = L\{t^4\}_{s=3}$$

$$= \frac{4!}{s^5} \Big|_{s=3}$$

$$= \frac{24}{3^5} //$$

2) Evaluate $\int_0^{\infty} e^{-3t} t \sin t dt = \int_0^{\infty} e^{-st} f(t) dt$

$s=3, f(t)=t \sin t$

$$\therefore \int_0^{\infty} e^{-3t} (t \sin t) dt = L\{t \sin t\}_{s=3}$$

$$= \frac{-d}{ds} L\{\sin t\}_{s=3}$$

$$= \frac{-d}{ds} \left[\frac{1}{s^2+1} \right]_{s=3}$$

$$= - \left(\frac{-2s}{(s^2+1)^2} \right) \Big|_{s=3}$$

$$= \frac{2s}{(s^2+1)^2} \Big|_{s=3}$$

$$= \frac{2(3)}{(9+1)^2} = \frac{6}{100} = \frac{3}{50}$$

3) Evaluate $\int_0^{\infty} e^{-2t} \frac{\sin 3t}{t} dt = \int_0^{\infty} e^{-st} f(t) dt.$

$$s = 2, f(t) = \frac{\sin 3t}{t}$$

$$\frac{1}{a} \tan^{-1}\left(\frac{y}{a}\right)$$

$$\therefore \int_0^{\infty} e^{-2t} \frac{\sin 3t}{t} dt = L \left\{ \frac{\sin 3t}{t} \right\}_{s=2}$$

$$= \int_0^{\infty} L \{ \sin 3t \} ds \Big|_{s=2}$$

$$= \int_0^{\infty} \left[L \{ \sin 3t \} \right] ds \Big|_{s=2}$$

$$= \frac{3}{s^2 + 9} \Big|_{s=2}$$

$$\pi \underline{3}$$

$$= \left[\int_0^{\infty} L \{ \sin 3t \} ds \right]_{s=2}$$

$$= \left[\int_0^{\infty} \frac{3}{s^2 + 3^2} ds \right]_{s=2}$$

$$= \left[\frac{1}{3} \tan^{-1}\left(\frac{s}{3}\right) \right]_{s=2}$$

$$= \left[\tan^{-1} \infty - \tan^{-1}\left(\frac{2}{3}\right) \right]_{s=2}$$

$$= \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{2}{3}\right) \right)_{s=2}$$

$$= \left(\cot^{-1}\left(\frac{3}{2}\right) \right)_{s=2}$$

$$= \cot^{-1}\left(\frac{2}{3}\right)$$

$$4) \int_0^{\infty} \frac{\cos 6t - \cos 4t}{t} dt$$

$$s=0, f(t) = \frac{\cos 6t - \cos 4t}{t}$$

$$\therefore \int_0^{\infty} \frac{\cos 6t - \cos 4t}{t} dt = L \left\{ \frac{\cos 6t - \cos 4t}{t} \right\}_{s=0}$$

$$= \left[\int_s^{\infty} [L\{\cos 6t\} - L\{\cos 4t\}] ds \right]_{s=0}$$

(By L.T of division by t)

$$= \left[\int_s^{\infty} \left(\frac{s}{s^2+36} - \frac{s}{s^2+16} \right) ds \right]_{s=0}$$

$$= \frac{1}{2} \left[\int_s^{\infty} \left(\frac{2s}{s^2+36} - \frac{2s}{s^2+16} \right) ds \right]_{s=0}$$

$$= \frac{1}{2} \left[(\log(s^2+36) - \log(s^2+16)) \right]_s^{\infty} \Big|_{s=0}$$

$$= \frac{1}{2} \left[\log \left(\frac{s^2+36}{s^2+16} \right) \right]_s^{\infty} \Big|_{s=0}$$

$$= \frac{1}{2} \left[\log \left(\frac{s^2(1+36/s^2)}{s^2(1+16/s^2)} \right) \right]_s^{\infty} \Big|_{s=0}$$

$$= \frac{1}{2} \left[\log \left(\frac{1+36/s^2}{1+16/s^2} \right) \right]_s^{\infty} \Big|_{s=0}$$

$$= \frac{1}{2} \left[\log \left(\frac{1+36/\infty}{1+16/\infty} \right) - \log \left(\frac{1+36/s^2}{1+16/s^2} \right) \right]_{s=0}$$

$$\log a^x$$

$$2 \log a$$

$$-\log\left(\frac{a}{b}\right) = \log\left(\frac{b}{a}\right)$$

$$= \left\{ \frac{1}{2} \left[\log(1) - \log\left(\frac{s^2+36}{s^2+16}\right) \right] \right\}_{s=0}$$

$$= \frac{1}{2} \left[0 - \log\left(\frac{s^2+36}{s^2+16}\right) \right]_{s=0}$$

$$= -\frac{1}{2} \log\left(\frac{s^2+36}{s^2+16}\right)_{s=0}$$

$$= -\frac{1}{2} \log \frac{16^4}{36^4}$$

$$= -\frac{1}{2} \log\left(\frac{2}{3}\right)^2 = \log\left(\frac{2}{3}\right)$$

Inverse L.T:

If $\bar{f}(s)$ is the L.T of ^{function} $f(t)$ then the function $f(t)$ is called the Inverse L.T of $\bar{f}(s)$ i.e., $L^{-1}\{\bar{f}(s)\} = f(t)$

Inverse L.T of standard function.

$f(t)$	$L\{f(t)\}$	$F(s)$	$L^{-1}\{F(s)\}$	$f(t)$
1	$L\{1\}$	$\frac{1}{s}$	$L^{-1}\{\frac{1}{s}\}$	1
e^{at}	$L\{e^{at}\}$	$\frac{1}{s-a}$	$L^{-1}\{\frac{1}{s-a}\}$	e^{at}
e^{-at}	$L\{e^{-at}\}$	$\frac{1}{s+a}$	$L^{-1}\{\frac{1}{s+a}\}$	e^{-at}
$\sin at$	$L\{\sin at\}$	$\frac{a}{s^2+a^2}$	$L^{-1}\{\frac{a}{s^2+a^2}\}$	$\frac{1}{a} \sin at$
$\sinh at$	$L\{\sinh at\}$	$\frac{a}{s^2-a^2}$	$L^{-1}\{\frac{1}{s^2-a^2}\}$	$\frac{1}{a} \sinh at$
$\cos at$	$L\{\cos at\}$	$\frac{s}{s^2+a^2}$	$L^{-1}\{\frac{s}{s^2+a^2}\}$	$\cos at$
$\cosh at$	$L\{\cosh at\}$	$\frac{s}{s^2-a^2}$	$L^{-1}\{\frac{s}{s^2-a^2}\}$	$\cosh at$
t	$L\{t\}$	$\frac{1}{s^2}$	$L^{-1}\{\frac{1}{s^2}\}$	t
t^n	$L\{t^n\}$	$\frac{\Gamma(n+1)}{s^{n+1}}$	$L^{-1}\{\frac{1}{s^{n+1}}\}$	$\frac{t^n}{\Gamma(n+1)}$

1) Find $L^{-1} \left\{ \frac{25-s}{s^2-4} \right\}$

$$\begin{aligned}
 &= L^{-1} \left\{ \frac{25}{s^2-2^2} - \frac{s}{s^2-2^2} \right\} \\
 &= L^{-1} \left\{ \frac{25}{s^2-2^2} \right\} - L^{-1} \left\{ \frac{s}{s^2-2^2} \right\} \\
 &= 25 L^{-1} \left\{ \frac{1}{s^2-2^2} \right\} - L^{-1} \left\{ \frac{s}{s^2-2^2} \right\} \\
 &= 25 \times \frac{1}{2} \sinh 2t - \cosh 2t \\
 &= \frac{25}{2} \sinh 2t - \cosh 2t
 \end{aligned}$$

2) Find $L^{-1} \left\{ \frac{s^2-3s+4}{s^3} \right\}$

$$\begin{aligned}
 &= L^{-1} \left\{ \frac{s^2}{s^3} \right\} - L^{-1} \left\{ \frac{3s}{s^3} \right\} + 4 L^{-1} \left\{ \frac{1}{s^3} \right\} \\
 &= L^{-1} \left\{ \frac{1}{s} \right\} - 3 L^{-1} \left\{ \frac{1}{s^2} \right\} + 4 L^{-1} \left\{ \frac{1}{s^3} \right\} \\
 &= 1 - 3t + 4 \times \frac{t^2}{2} \\
 &= 1 - 3t + 2t^2
 \end{aligned}$$

3) $L^{-1} \left\{ \frac{3s-8}{4s^2+25} \right\}$

$$\begin{aligned}
 &= L^{-1} \left\{ \frac{3s}{4s^2+25} - \frac{8}{4s^2+25} \right\} \\
 &= 3 L^{-1} \left\{ \frac{s}{4s^2+25} \right\} - 8 L^{-1} \left\{ \frac{1}{4s^2+25} \right\} \\
 &= \frac{3}{4} L^{-1} \left\{ \frac{s}{s^2+25/4} \right\} - \frac{8}{4} L^{-1} \left\{ \frac{1}{s^2+25/4} \right\}
 \end{aligned}$$

$$= \frac{3}{4} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + (\frac{5}{2})^2} \right\} - 2 \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + (\frac{5}{2})^2} \right\}$$

$$= \frac{3}{4} \cos\left(\frac{5}{2}t\right) - 2 \times \frac{1}{5/2} \sin\left(\frac{5}{2}t\right)$$

$$= \frac{3}{4} \cos\left(\frac{5}{2}t\right) - \frac{4}{5} \sin\left(\frac{5}{2}t\right)$$

4) Find $\mathcal{L}^{-1} \left\{ \frac{s}{2s^2 + 8} \right\}$

$$= \mathcal{L}^{-1} \left\{ \frac{s}{2(s^2 + 4)} \right\}$$

$$= \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 2^2} \right\}$$

$$= \frac{1}{2} \cos 2t$$

Inverse L.T using partial functions:

Suppose $\bar{f}(s) = \frac{\bar{P}(s)}{\bar{Q}(s)}$ where,

$\bar{P}(s)$ and $\bar{Q}(s)$ are polynomials of s , such that the degree of $\bar{P}(s)$ is ~~of form~~ ^{less than} degree of $\bar{Q}(s)$.

Case-(i):

when $\bar{Q}(s)$ contains ^{product of} n distinct linear factors.

$$\bar{Q}(s) = (s+a_1)(s+a_2) \dots (s+a_n)$$

then $\bar{f}(s) = \frac{A_1}{s+a_1} + \frac{A_2}{s+a_2} + \dots + \frac{A_n}{s+a_n}$

case-(ii):

When factors are linear and equal i.e.,

$$\bar{Q}(s) = (s+a)(s+b)^n$$

Then

$$\bar{f}(s) = \frac{A_1}{s+a} + \frac{B_1}{s+b} + \frac{B_2}{(s+b)^2} + \dots + \frac{B_n}{(s+b)^n}$$

case-(iii):

When factors are quadratic and distinct i.e.,

$$\bar{Q}(s) = (s^2+as+b)(s^2+cs+d)$$

then

$$\bar{f}(s) = \frac{As+B}{s^2+as+b} + \frac{Cs+D}{s^2+cs+d}$$

case-(iv):

When factors are quadratic and equal i.e.,

$$\bar{Q}(s) = (s^2+as+b)(s^2+cs+d)^n$$

then

$$\bar{f}(s) = \frac{As+B}{s^2+as+b} + \frac{C_1s+D_1}{(s^2+cs+d)} + \frac{C_2s+D_2}{(s^2+cs+d)^2} + \dots + \frac{C_ns+D_n}{(s^2+cs+d)^n}$$

1) Find $L^{-1} \left\{ \frac{2s+1}{s(s+1)} \right\}$

$$\text{Let } \bar{f}(s) = \frac{2s+1}{s(s+1)}$$

$$= \frac{2s+1}{(s+0)(s+1)}$$

$$\Rightarrow \frac{2s+1}{(s+0)(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$\Rightarrow 2s+1 = A(s+1) + Bs \quad \text{--- (1)}$$

put $s=0$ in ①

$$\boxed{1 = A}$$

put $s = -1$ in ①

$$-1 = -B \Rightarrow \boxed{B = 1}$$

$$\therefore \bar{f}(s) = \frac{1}{s} + \frac{1}{s+1}$$

Apply I.L.T

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s(s+1)}\right\} = 1 + e^{-t}$$

2) Find $\mathcal{L}^{-1}\left\{\frac{s+2}{s(s+1)(s+3)}\right\}$

Let $\bar{f}(s) = \frac{s+2}{s(s+1)(s+3)}$ (factors are linear & distinct)

$$\Rightarrow \bar{f}(s) = \frac{s+2}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$s+2 = A(s+1)(s+3) + Bs(s+3) + Cs(s+1) \quad \text{--- ①}$$

$$\text{put } s=0 \text{ in ①, } 2 = 3A \Rightarrow \boxed{A = \frac{2}{3}}$$

$$\text{put } s = -1 \text{ in ①, } 1 = -2B \Rightarrow \boxed{B = -\frac{1}{2}}$$

$$\text{put } s = -3 \text{ in ①, } -1 = 6C \Rightarrow \boxed{C = -\frac{1}{6}}$$

$$\therefore \frac{s+2}{s(s+1)(s+3)} = \frac{2}{3s} - \frac{1}{2(s+1)} - \frac{1}{6(s+3)}$$

Apply I.L.T

$$\mathcal{L}^{-1} \left\{ \frac{s+2}{s(s+1)(s+2)} \right\} = \frac{2}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} - \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} - \frac{1}{6} \mathcal{L}^{-1} \left\{ \frac{1}{s+3} \right\}$$

$$= \frac{2}{3} (1) - \frac{1}{2} e^{-t} - \frac{1}{6} e^{-3t}$$

$$= \frac{2}{3} - \frac{1}{2} e^{-t} - \frac{1}{6} e^{-3t}$$

$$3) \mathcal{L}^{-1} \left\{ \frac{s}{(s^2+1)(s^2+4)} \right\}$$

Factors are quadratic & distinct

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$s = (A+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$s = As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D \quad \text{--- (1)}$$

Comparing coefficients on B. S

$$i) s^3: A+C=0 \quad \text{--- (a)}$$

$$ii) s^2: B+D=0 \quad \text{--- (b)}$$

$$s: 4A+C=1 \quad \text{--- (c)}$$

$$\text{constant: } 4B+D=0 \quad \text{--- (d)}$$

$$A = \frac{1}{3}, C = -\frac{1}{3}, B = 0, D = 0$$

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{\frac{1}{3}s+0}{s^2+1} + \frac{-\frac{1}{3}s+0}{s^2+4}$$

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{1}{3} \left[\frac{s}{s^2+1} - \frac{s}{s^2+4} \right]$$

Apply I.L.T

$$\mathcal{L}^{-1} \left\{ \frac{s}{(s^2+1)(s^2+4)} \right\} = \frac{1}{3} \left(\mathcal{L}^{-1} \left\{ \frac{s}{s^2+1} \right\} - \mathcal{L}^{-1} \left\{ \frac{s}{s^2+2^2} \right\} \right)$$

$$= \frac{1}{3} [\cos t - \cos 2t]$$

ILT of First shifting Theorem:

$$\text{If } \mathcal{L}^{-1}\{\bar{f}(s)\} = f(t) \text{ then } \mathcal{L}^{-1}\{\bar{f}(s-a)\} = e^{at} f(t) \\ = e^{at} \mathcal{L}^{-1}\{\bar{f}(s)\}$$

Note:

$$\mathcal{L}^{-1}\{\bar{f}(s+a)\} = e^{-at} f(t) \\ = e^{-at} \mathcal{L}^{-1}\{\bar{f}(s)\}.$$

*) Find $\mathcal{L}^{-1}\left\{\frac{s}{(s+3)(s+1)^2}\right\}$

*) Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\}.$

1) Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\} = \mathcal{L}^{-1}\{\bar{f}(s+2)\}$
 $\bar{f}(s) = \frac{1}{s^3}$

By I.L.T of F.S.T

$$\mathcal{L}^{-1}\{\bar{f}(s+2)\} = e^{-2t} \mathcal{L}^{-1}\{\bar{f}(s)\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\} = e^{-2t} \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= e^{-2t} \frac{t^2}{2!}$$

$$= \frac{e^{-2t} t^2}{2} \left[\because \mathcal{L}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!} \right]$$

2) Find $L^{-1} \left\{ \frac{s}{(s+3)(s+1)^2} \right\}$

Given $\bar{f}(s) = \frac{s}{(s+3)(s+1)^2} = \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$

$\Rightarrow s = A(s+1)^2 + B(s+3)(s+1) + C(s+3)$ ①

put $s = -3$ in ①

$-3 = 4A \Rightarrow \boxed{A = -\frac{3}{4}}$

put $s = -1$ in ①

$-1 = 2C \Rightarrow \boxed{C = -\frac{1}{2}}$

compare the coeff. of s^2

$0 = A + B$

$\Rightarrow 0 = -\frac{3}{4} + B \Rightarrow \boxed{B = \frac{3}{4}}$

$\therefore \frac{s}{(s+3)(s+1)^2} = \frac{-3}{4(s+3)} + \frac{3}{4(s+1)} - \frac{1}{2(s+1)^2}$

Apply I. L. T

$L^{-1} \left\{ \frac{s}{(s+3)(s+1)^2} \right\} = -\frac{3}{4} L^{-1} \left\{ \frac{1}{s+3} \right\} + \frac{3}{4} L^{-1} \left\{ \frac{1}{s+1} \right\} - \frac{1}{2} L^{-1} \left\{ \frac{1}{(s+1)^2} \right\}$

$= -\frac{3}{4} e^{-3t} + \frac{3}{4} e^{-t} - \frac{1}{2} e^{-t} L^{-1} \left\{ \frac{1}{s^2} \right\}$

$= \frac{3}{4} [e^{-t} - e^{-3t}] - \frac{1}{2} e^{-t} t$

3) Find $\mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 10s + 26} \right\}$

Given that

$$\bar{f}(s) = \frac{s}{s^2 - 10s + 26} = \frac{s}{s^2 - 2(s)5 + 25 + 1}$$

$$= \frac{s - 5 + 5}{(s - 5)^2 + 1^2}$$

$$= \frac{(s - 5)}{(s - 5)^2 + 1^2} + \frac{5}{(s - 5)^2 + 1^2}$$

$$\mathcal{L}^{-1} \left\{ \frac{s}{s^2 - 10s + 26} \right\} = \mathcal{L}^{-1} \left\{ \frac{s - 5}{(s - 5)^2 + 1^2} + \frac{5}{(s - 5)^2 + 1^2} \right\} = \mathcal{L}^{-1} \{ \bar{f}(s - 5) \}$$

$$= e^{5t} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 1} + \frac{5}{s^2 + 1} \right\} \text{ (By I.L.T of F.S.T)}$$

$$= e^{5t} \left[\mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 1^2} \right\} + 5 \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1^2} \right\} \right]$$

$$= e^{5t} (\cos t + 5 \sin t)$$

4) $\mathcal{L}^{-1} \left\{ \frac{s + 2}{s^2 + 4s + 13} \right\}$

$$\text{Let } \bar{f}(s) = \frac{s + 2}{s^2 + 4s + 13} = \frac{s + 2}{s^2 + 2(2s) + 4 + 9}$$

$$= \frac{s + 2}{(s + 2)^2 + 3^2}$$

$$\mathcal{L}^{-1} \left\{ \frac{s + 2}{s^2 + 4s + 13} \right\} = \mathcal{L}^{-1} \left\{ \frac{(s + 2)}{(s + 2)^2 + 3^2} \right\} = \mathcal{L}^{-1} \{ \bar{f}(s + 2) \}$$

$$= e^{-2t} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 3^2} \right\}$$

$$= e^{-2t} \cos 3t //$$

$$5) \quad \mathcal{L}^{-1} \left\{ \frac{2s+12}{s^2+6s+13} \right\}$$

$$b=3$$

$$a=4$$

$$\text{let } \left\{ \frac{2s+12}{s^2+6s+13} \right\}$$

$$\bar{p}(s) = \frac{2s+12}{s^2+6s+13} = \frac{2s+12}{s^2+2(3s)+9+4}$$

$$= \frac{2s+6+6}{(s+3)^2+4}$$

$$= \frac{2(s+3)+6}{(s+3)^2+2^2}$$

$$\mathcal{L}^{-1} \left\{ \frac{2s+12}{s^2+6s+13} \right\} = \mathcal{L}^{-1} \left\{ \frac{2(s+3)+6}{(s+3)^2+2^2} \right\} = \mathcal{L}^{-1} \left\{ \bar{p}(s+3) \right\}$$

$$= e^{-3t} \mathcal{L}^{-1} \left\{ \frac{2s+6}{s^2+2^2} \right\}$$

$$= e^{-3t} \left[2 \mathcal{L}^{-1} \left\{ \frac{s}{s^2+2^2} \right\} + 6 \mathcal{L}^{-1} \left\{ \frac{1}{s^2+2^2} \right\} \right]$$

$$= e^{-3t} \left[2 \cos 2t + 6 \frac{1}{2} \sin 2t \right]$$

$$= e^{-3t} [2 \cos 2t + 3 \sin 2t] //$$

I.L.T of Second-shifting Theorem:

If $\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$ then $\mathcal{L}^{-1}\{e^{-as}\bar{f}(s)\} = g(t)$
$$= \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

i) Find $\mathcal{L}^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$$

$$= \sin t + \mathcal{L}^{-1}\{e^{-as}\bar{f}(s)\} \quad \text{where } \bar{f}(s) = \frac{1}{s^2+1}$$

$$= \sin t + g(t) \quad \begin{aligned} \mathcal{L}^{-1}\{\bar{f}(s)\} &= \sin t \\ &= f(t) \end{aligned}$$

$$\text{where } g(t) = \begin{cases} \sin(t-\pi), & t > \pi \\ 0, & t < \pi \end{cases} \quad a = \pi$$

Inverse L-T of derivatives:

If $\bar{L}^{-1}\{\bar{f}(s)\} = f(t)$, then

$$\bar{L}^{-1}\left\{\frac{d^n}{ds^n}\bar{f}(s)\right\} = (-1)^n t^n f(t)$$

$$\text{or} \\ (-1)^n t^n \bar{L}^{-1}\{\bar{f}(s)\}$$

Note:

I.L.T of derivatives is applicable for

$\log, \cot^{-1}, \tan^{-1}$

1) Find $\bar{L}^{-1}\left\{\log\left(\frac{s+1}{s+2}\right)\right\}$

$$\text{Let } \bar{f}(s) = \log\left(\frac{s+1}{s+2}\right)$$

$$\Rightarrow \bar{f}(s) = \log(s+1) - \log(s+2) \quad \text{--- (i)}$$

Diff (i) w.r.t 's'

$$\frac{d}{ds} \bar{f}(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Apply I.L.T

$$\bar{L}^{-1}\left\{\frac{d}{ds} \bar{f}(s)\right\} = \bar{L}^{-1}\left\{\frac{1}{s+1}\right\} - \bar{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\Rightarrow (-1)^1 t^1 f(t) = e^{-t} - e^{-2t}$$

By I.L.T of derivative

$$\Rightarrow -t \bar{L}^{-1}\{\bar{f}(s)\} = e^{-t} - e^{-2t}$$

$$\Rightarrow \bar{L}^{-1}\left\{\log\left(\frac{s+1}{s+2}\right)\right\} = \frac{e^{-2t} - e^{-t}}{t}$$

2) Find $\mathcal{L}^{-1}\{\tan^{-1}(s+1)\}$

$$\bar{f}(s) = \tan^{-1}(s+1)$$

Diff w.r.t 's'

$$\frac{d}{ds} \bar{f}(s) = \frac{1}{1+(s+1)^2}$$

Apply I.L.T

$$\mathcal{L}^{-1}\left\{\frac{d}{ds} \bar{f}(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{1+(s+1)^2}\right\} [\mathcal{L}^{-1}(\bar{f}(s+1))]$$

$$\rightarrow t f(t) = e^{-t} \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \quad (\text{By I.L.T of F.S.T})$$

$$\rightarrow t \mathcal{L}^{-1}\{\bar{f}(s)\} = e^{-t} \sin t$$

$$\mathcal{L}^{-1}\{\tan^{-1}(s+1)\} = \frac{-e^{-t} \sin t}{t}$$

3) Find $\mathcal{L}^{-1}\left\{\log\left(\sqrt{\frac{s^2+1}{s^2}}\right)\right\}$

$$\text{Let } \bar{f}(s) = \log\left(\frac{\sqrt{s^2+1}}{s}\right)$$

$$\Rightarrow \bar{f}(s) = \log(\sqrt{s^2+1}) - \log s$$

Diff w.r.t 's'

$$\frac{d}{ds} \bar{f}(s) = \frac{1}{\sqrt{s^2+1}} \times \frac{1}{2\sqrt{s^2+1}} (2s) - \frac{1}{s} (1)$$

$$\Rightarrow \frac{d}{ds} \bar{f}(s) = \frac{s}{s^2+1} - \frac{1}{s}$$

Apply I.L.T

$$\mathcal{L}^{-1}\left\{\frac{d}{ds} \bar{f}(s)\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$\Rightarrow (-D + f(t)) = \cos t - 1$$

$$\Rightarrow -t \mathcal{L}^{-1}\{\bar{f}(s)\} = \cos t - 1$$

$$\Rightarrow \mathcal{L}^{-1}\{\bar{f}(s)\} = \frac{1 - \cos t}{t} //$$

$$4) \mathcal{L}^{-1}\{\cot^{-1}(s)\}$$

$$\text{Let } \bar{f}(s) = \cot^{-1}(s)$$

diff w.r.t 's'

$$\frac{d}{ds}(\bar{f}(s)) = \frac{-1}{1+s^2}$$

Apply I. L. T

$$\mathcal{L}^{-1}\left\{\frac{d}{ds}(\bar{f}(s))\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{1+s^2}\right\}$$

$$(-D + f(t)) = -\mathcal{L}^{-1}\left\{\frac{1}{1+s^2}\right\}$$

$$-t f(t) = -\sin t$$

$$-t \mathcal{L}^{-1}\{\bar{f}(s)\} = -\sin t$$

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = \frac{\sin t}{t}$$

$$5) \quad L^{-1} \{ \cot^{-1}(\frac{2}{s^2}) \}$$

$$\bar{f}(s) = \cot^{-1}(\frac{2}{s^2})$$

diff. w.r.t 's'

$$\frac{d}{ds} (\bar{f}(s)) = \frac{-1}{1 + (\frac{2}{s^2})^2}$$

Inverse L.T of Integrals :

If $\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$ then

$$\mathcal{L}^{-1}\left\{\int_s^{\infty} \bar{f}(s) ds\right\} = \frac{f(t)}{t} = \frac{\mathcal{L}^{-1}\{\bar{f}(s)\}}{t}$$

Note :

$$\boxed{\mathcal{L}^{-1}\{\bar{f}(s)\} = t \mathcal{L}^{-1}\left\{\int_s^{\infty} \bar{f}(s) ds\right\}}$$

1. Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$

$$\text{Let } \bar{f}(s) = \frac{1}{(s+1)^2}$$

By I.L.T of integrals, we have

$$\mathcal{L}^{-1}\left\{\int_s^{\infty} \bar{f}(s) ds\right\} = \frac{\mathcal{L}^{-1}\{\bar{f}(s)\}}{t}$$

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = t \mathcal{L}^{-1}\left\{\int_s^{\infty} \frac{1}{(s+1)^2} ds\right\}$$

$$= t \mathcal{L}^{-1}\left\{\left[\frac{-1}{s+1}\right]_s^{\infty}\right\}$$

$$= t \mathcal{L}^{-1}\left\{-\left[0 - \frac{1}{s+1}\right]\right\}$$

$$= t \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= t e^{-t}$$

2. Find $\mathcal{L}^{-1} \left\{ \frac{2s}{(s^2+2)^2} \right\}$

Let $\bar{f}(s) = \frac{2s}{(s^2+2)^2}$

By I.L.T of Integrals

$$\mathcal{L}^{-1} \{ \bar{f}(s) \} = t \mathcal{L}^{-1} \left\{ \int_s^\infty \bar{f}(s) ds \right\}$$

$$\Rightarrow \mathcal{L}^{-1} \left\{ \frac{2s}{(s^2+2)^2} \right\} = t \mathcal{L}^{-1} \left\{ \int_s^\infty \frac{2s}{(s^2+2)^2} ds \right\}$$

$$= t \mathcal{L}^{-1} \left\{ \int_s^\infty 2s(s^2+2)^{-2} ds \right\}$$

$$= t \mathcal{L}^{-1} \left\{ \left[\frac{(s^2+2)^{-2+1}}{-2+1} \right]_s^\infty \right\}$$

$$= t \mathcal{L}^{-1} \left\{ - \left[\frac{1}{s^2+2} \right]_s^\infty \right\}$$

$$= t \mathcal{L}^{-1} \left\{ - \left(0 - \frac{1}{s^2+2} \right) \right\}$$

$$= t \mathcal{L}^{-1} \left\{ \frac{1}{s^2+(\sqrt{2})^2} \right\}$$

$$= t \times \frac{1}{\sqrt{2}} \sin \sqrt{2}t //$$

3. Find $\mathcal{L}^{-1} \left\{ \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1} \right) ds \right\}$

$$\mathcal{L}^{-1} \left\{ \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1} \right) ds \right\} = \mathcal{L}^{-1} \left\{ \int_s^\infty \bar{f}(s) ds \right\}$$

Let $\bar{f}(s) = \frac{1}{s} - \frac{1}{s-1}$

By I.L.T of Integrals

$$\mathcal{L}^{-1} \left\{ \int_s^\infty \bar{f}(s) ds \right\} = \frac{\mathcal{L}^{-1} \{ \bar{f}(s) \}}{t}$$

$$\Rightarrow \mathcal{L}^{-1} \left\{ \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1} \right) ds \right\} = \frac{\mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{1}{s-1} \right\}}{t}$$

$$= \frac{1}{t} \left[\mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} \right]$$

$$\mathcal{L}^{-1} \left\{ \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1} \right) ds \right\} = \frac{1}{t} (1 - e^t).$$

4. Find $\mathcal{L}^{-1} \left\{ \int_s^\infty \log \left(\frac{s-1}{s+1} \right) ds \right\} = \mathcal{L}^{-1} \left\{ \int_s^\infty \bar{f}(s) ds \right\}$

Let $\bar{f}(s) = \log \left(\frac{s-1}{s+1} \right)$

$\Rightarrow \bar{f}(s) = \log(s-1) - \log(s+1)$

Diff w.r.t 's'

$$\frac{d}{ds} \bar{f}(s) = \frac{1}{s-1} - \frac{1}{s+1}$$

Apply I.L.T

$$\mathcal{L}^{-1} \left\{ \frac{d}{ds} \bar{f}(s) \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\}$$

By I.L.T of first derivative

$$(-1)^1 t f(t) = e^t - e^{-t}$$

$$\Rightarrow f(t) = \frac{1}{t} (e^{-t} - e^t) \quad \text{--- (1)}$$

By I.L.T of Integrals

$$\mathcal{L}^{-1} \left\{ \int_s^\infty \bar{f}(s) ds \right\} = \frac{f(t)}{t}$$

$$\Rightarrow \mathcal{L}^{-1} \left\{ \int_s^\infty \log \left(\frac{s-1}{s+1} \right) ds \right\} = \frac{1}{t^2} (e^{-t} - e^t) //$$

I.L.T of multiplication by 's':

If $\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$ where $f(t)$ is a piecewise continuous and of exponential order and $f(0) = 0$,

Then $\mathcal{L}^{-1}\{s \bar{f}(s)\} = f'(t)$ or $\frac{d}{dt} f(t)$ or $\frac{d}{dt} \mathcal{L}^{-1}\{\bar{f}(s)\}$

$$\mathcal{L}^{-1}\{s^2 \bar{f}(s)\} = f''(t)$$

$$\mathcal{L}^{-1}\{s^3 \bar{f}(s)\} = f'''(t)$$

$$\mathcal{L}^{-1}\{s^n \bar{f}(s)\} = f^{(n)}(t)$$

1) Find $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^4}\right\}$

$$\text{Let } \bar{f}(s) = \frac{1}{(s-2)^4}$$

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\} = \mathcal{L}^{-1}\{\bar{f}(s-2)\}$$

$$= e^{2t} \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} \quad (\text{By I.L.T of F.S.T})$$

$$= e^{2t} \frac{t^3}{3!}$$

$$= \frac{e^{2t} t^3}{6} = f(t) \quad \text{--- ①}$$

By I.L.T of multiplication by 's'

$$\mathcal{L}^{-1}\{s \bar{f}(s)\} = \frac{d}{dt} f(t)$$

$$\mathcal{L}^{-1}\left\{s \frac{1}{(s-2)^4}\right\} = \frac{d}{dt} \left(\frac{e^{2t} t^3}{6}\right)$$

$$= \frac{1}{6} \left[e^{2t} (3t^2) + (2e^{2t}) t^3 \right]$$

$$= \frac{t^2 e^{2t}}{6} [3 + 2t]$$

2. Find $L^{-1}\left\{\frac{s^2}{(s^2+4)}\right\} = L^{-1}\{s^2\bar{f}(s)\}$

Let $\bar{f}(s) = \frac{1}{s^2+4}$

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

$$= \frac{1}{2} \sin 2t = f(t)$$

By I.L.T of multiplication by 's'

$$L^{-1}\{s^2\bar{f}(s)\} = \frac{d^2}{dt^2} f(t)$$

$$L^{-1}\left\{\frac{s^2}{s^2+4}\right\} = \frac{d^2}{dt^2} \left[\frac{1}{2} \sin 2t\right]$$

$$= \frac{1}{2} \frac{d}{dt} \left[\frac{d}{dt} \sin 2t\right]$$

$$= \frac{1}{2} \frac{d}{dt} [2 \cos 2t]$$

$$= \frac{d}{dt} (\cos 2t)$$

$$= -2 \sin 2t$$

3. Find $L^{-1}\left\{\frac{2s^3}{(s^2+4)^2}\right\}$

$$L^{-1}\left\{s^2 \left[\frac{2s}{(s^2+4)^2}\right]\right\} = L^{-1}\{s^2\bar{f}(s)\}$$

Let $\bar{f}(s) = \frac{2s}{(s^2+4)^2}$

$$L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{2s}{(s^2+4)^2}\right\} \quad (\text{I.L.T of Integrals})$$

$$= t L^{-1}\left\{\int_0^\infty \frac{2s}{(s^2+4)^2} ds\right\}$$

$$= t L^{-1}\left\{\int_0^\infty (2s)(s^2+4)^{-2} ds\right\}$$

$$= t L^{-1} \left\{ \left[\frac{(s^2+4)^{-2+1}}{-2+1} \right]_s^\infty \right\}$$

$$= t L^{-1} \left\{ - \left[\frac{1}{s^2+2^2} \right]_s^\infty \right\}$$

$$= t L^{-1} \left\{ - \left[0 - \frac{1}{s^2+2^2} \right] \right\}$$

$$= t L^{-1} \left\{ \frac{1}{s^2+2^2} \right\}$$

$$= \frac{t}{2} \sin 2t = f(t)$$

By I.L.T multiplication by s^2

$$L^{-1} \{ s^2 \bar{f}(s) \} = \frac{d^2}{dt^2} f(t)$$

$$\Rightarrow L^{-1} \left\{ \frac{2s^3}{(s^2+4)^2} \right\} = \frac{d^2}{dt^2} \left(\frac{t}{2} \sin 2t \right)$$

$$= \frac{1}{2} \frac{d}{dt} \left[t(2 \cos 2t) + \sin 2t \right]$$

$$= \frac{1}{2} \left[2(t(-2 \sin 2t) + 2 \cos 2t + 2 \cos 2t) \right]$$

$$= \frac{1}{2} [-4t \sin 2t + 4 \cos 2t]$$

$$= 2(\cos 2t - t \sin 2t),$$

4) Find $L^{-1} \left\{ s \int_0^\infty \log \left(\frac{s+2}{s+3} \right) ds \right\} = L^{-1} \left\{ s \int_0^\infty \bar{f}(s) ds \right\}$

Let $\bar{f}(s) = \log \left(\frac{s+2}{s+3} \right)$

I.L.T of division by s :

$$\text{If } \mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$$

$$\begin{aligned} \text{then } \mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} &= \int_0^t f(t) dt \\ &= \int_0^t \mathcal{L}^{-1}\{\bar{f}(s)\} dt \end{aligned}$$

1) Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s+a)}\right\} = \mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\}$

$$\text{Here } \bar{f}(s) = \frac{1}{s+a}$$

$$\text{Now } \mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at} = f(t)$$

By I. L. T of division by ' s '

$$\mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t) dt$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+a)}\right\} = \int_0^t e^{-at} dt$$

$$= \left[\frac{e^{-at}}{-a}\right]_0^t$$

$$= -\frac{1}{a} [e^{-at} - e^0]$$

$$= \frac{1}{a} [1 - e^{-at}]$$

2) Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\}$

$$\text{Here } \bar{f}(s) = \frac{1}{s^2+2s+2}$$

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t} \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \quad (\text{By I. L. T of F. S. T})$$

$$= e^{-t} \sin t = f(t)$$

By I.L.T of division by 's'

$$\mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t) dt$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2st+1)}\right\} = \int_0^t e^{-t} \sin t dt$$

$$= \left[\frac{e^{-t}}{(-1)^{\frac{1}{2}}} (-1)^{\frac{1}{2}} (\sin t - \cos t) \right]_0^t$$

$$= \frac{e^{-t}}{2} (-\sin t - \cos t) - \frac{e^0}{2} (-\sin 0 - \cos 0)$$

$$= -\frac{e^{-t}}{2} (\sin t + \cos t) + \frac{1}{2}$$

$$= \frac{1}{2} [1 - e^{-t} (\sin t + \cos t)]$$

3) Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = \mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\}$

where $\bar{f}(s) = \frac{1}{s^2+a^2}$

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$= \frac{1}{a} \sin at = f(t)$$

By I.L.T of division by 's²'

$$\mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \int_0^t f(t) dt dt$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = \int_0^t \int_0^t \frac{1}{a} \sin at dt dt$$

$$= \frac{1}{a} \left[\int_0^t \left(-\frac{\cos at}{a} \right) dt \right]$$

$$= \frac{-1}{a^2} \int_0^t (\cos at - \cos 0) dt$$

$$\begin{aligned}
&= \frac{-1}{a^2} \int_0^t (\cos at - 1) dt \\
&= \frac{-1}{a^2} \left[\frac{\sin at}{a} - t \right]_0^t \\
&= \frac{-1}{a^2} \left[\left(\frac{\sin at}{a} - t \right) - \left(\frac{\sin 0}{a} - 0 \right) \right] \\
&= \frac{-1}{a^2} \left[\frac{\sin at}{a} - t \right] \\
&= \frac{1}{a^2} \left[t - \frac{\sin at}{a} \right]
\end{aligned}$$

Convolution product:

If $f(t)$ & $g(t)$ are defined for $t \geq 0$, then the convolution product of $f(t)$ & $g(t)$ is denoted by $f(t) * g(t)$ and is defined as

$$f(t) * g(t) = \int_0^t f(u)g(t-u) du$$

Properties:

1) Convolution product satisfies commutative property

i.e., $f(t) * g(t) = g(t) * f(t)$

2) Convolution product satisfies associative property

i.e., $f(t) * [g(t) * h(t)] = [f(t) * g(t)] * h(t)$

3) $f(t) * 0 = 0$

4) $f(t) * 1 \neq f(t)$

Convolution Theorem:

If $\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$ & $\mathcal{L}^{-1}\{\bar{g}(s)\} = g(t)$ then

$$\mathcal{L}^{-1}\{\bar{f}(s) \cdot \bar{g}(s)\} = f(t) * g(t)$$

$$= \int_0^t f(u) g(t-u) du$$

1) Find $\mathcal{L}^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+a} \cdot \frac{1}{s+b}\right\}$

Let $\bar{f}(s) = \frac{1}{s+a} \Rightarrow \mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at} = f(t)$

& $\bar{g}(s) = \frac{1}{s+b} \Rightarrow \mathcal{L}^{-1}\{\bar{g}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+b}\right\} = e^{-bt} = g(t)$

By convolution theorem,

$$\mathcal{L}^{-1}\{\bar{f}(s) \bar{g}(s)\} = f(t) * g(t) = \int_0^t f(u) g(t-u) du$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s+a} \cdot \frac{1}{s+b}\right\} = \int_0^t e^{-au} \cdot e^{-b(t-u)} du$$

$$= \int_0^t e^{-au} \cdot e^{-bt} \cdot e^{bu} du$$

$$= e^{-bt} \int_0^t e^{(b-a)u} du$$

$$= e^{-bt} \left[\frac{e^{(b-a)u}}{b-a} \right]_0^t$$

$$= \frac{e^{-bt}}{b-a} \left[e^{(b-a)t} - e^{(b-a)0} \right]$$

$$= \frac{e^{-bt}}{b-a} \left[e^{bt} \cdot e^{-at} - 1 \right]$$

$$= \frac{1}{b-a} \left[e^{-at} - e^{-bt} \right]$$

$$2) \text{ Find } \mathcal{L}^{-1} \left\{ \frac{1}{s(s^2+4)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s^2+4} \cdot \frac{1}{s} \right\}$$

$$\text{Let } \bar{f}(s) = \frac{1}{s^2+2^2} \Rightarrow \mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

$$= \frac{1}{2} \sin 2t = f(t)$$

$$\bar{g}(s) = \frac{1}{s} \Rightarrow \mathcal{L}^{-1}\{\bar{g}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 = g(t)$$

By convolution theorem

$$\mathcal{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} = f(t) * g(t) = \int_0^t f(u)g(t-u)du$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s^2+4}\right\} = \int_0^t \frac{1}{2} \sin 2u du$$

$$= \frac{1}{2} \int_0^t \sin 2u du$$

$$= \frac{1}{2} \left[-\frac{\cos 2u}{2} \right]_0^t$$

$$= -\frac{1}{4} [\cos 2t - \cos 0]$$

$$= -\frac{1}{4} [\cos 2t - 1]$$

$$= \frac{1}{4} [1 - \cos 2t]$$

3) Apply convolution theorem to evaluate.

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s+2)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s+2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2} \cdot \frac{1}{s-2}\right\}$$

$$\text{Let } \bar{f}(s) = \frac{1}{(s+2)^2} \Rightarrow \mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$= e^{-2t} \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \quad (\text{By F.S.T})$$

$$= e^{-2t} t = f(t)$$

$$\bar{g}(s) = \frac{1}{s-2} \Rightarrow \mathcal{L}^{-1}\{\bar{g}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t} = g(t)$$

By convolution theorem $\mathcal{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} =$

$$= \int_0^t f(u)g(t-u)du$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\} = \int_0^t e^{-2u} u e^{2(t-u)} du$$

$$= \int_0^t u e^{-2u} \cdot e^{2t} \cdot e^{-2u} du$$

$$= e^{2t} \int_0^t u e^{-4u} du$$

$$= e^{2t} \left[u \frac{e^{-4u}}{-4} - \frac{e^{-4u}}{(4)^2} \right]_0^t$$

$$= e^{2t} \left[-\frac{t}{4} e^{-4t} - \frac{1}{16} e^{-4t} - \left(0 - \frac{1}{16} e^0 \right) \right]$$

4) Find $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1} \cdot \frac{1}{s^2+4}\right\}$$

$$\text{Let } \bar{f}(s) = \frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t = f(t)$$

$$\bar{g}(s) = \frac{1}{s^2+2^2} \Rightarrow \mathcal{L}^{-1}\{\bar{g}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} \\ = \frac{1}{2} \sin 2t = g(t)$$

By convolution Theorem

$$\mathcal{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} = \int_0^t f(u)g(t-u)du$$

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{s}{s^2+1} \cdot \frac{1}{s^2+4}\right\} &= \int_0^t \cos u \cdot \frac{1}{2} \sin 2(t-u) du \\ &= \frac{1}{2} \int_0^t \cos u \cdot \sin(2t-2u) du \\ &= \frac{1}{4} \int_0^t (\sin(2t-2u) - 2\cos u \cdot \sin(2t-2u)) du \\ &= \frac{1}{4} \int_0^t (\sin(u+2t-2u) - \sin(u-2t+2u)) du \\ &= \frac{1}{4} \int_0^t (\sin(2t-u) - \sin(3u-2t)) du \\ &= \frac{1}{4} \left[\int_0^t \sin(2t-u) du - \int_0^t \sin(3u-2t) du \right] \end{aligned}$$

$$= \frac{1}{4} \left[\left(\frac{\cos(2t-u)}{-1} \right)_0^t - \left(\frac{-\cos(3u-2t)}{3} \right)_0^t \right]$$

$$= \frac{1}{4} \left[\left(\cos(2t-u) \right)_0^t + \left(\frac{\cos(3u-2t)}{3} \right)_0^t \right]$$

$$= \frac{1}{4} \left[\cos t - \cos 2t + \frac{\cos t + \cos 2t}{3} \right]$$

$$= \frac{1}{4} \left[\frac{3\cos t - 3\cos 2t + \cos t + \cos 2t}{3} \right]$$

$$= \frac{1}{12} [4\cos t - 2\cos 2t]$$

$$= \frac{1}{6} [\cos t - \cos 2t]$$

$$5) \mathcal{L}^{-1} \left\{ \frac{s^2}{(s^2+a^2)^2} \right\} = \mathcal{L}^{-1} \left\{ \frac{s}{s^2+a^2} - \frac{s}{s^2+a^2} \right\}$$

Let

$$\bar{f}(s) = \frac{s}{s^2+a^2} \Rightarrow \mathcal{L}^{-1} \left\{ \frac{s}{s^2+a^2} \right\} = \cos at = f(t)$$

$$\bar{g}(s) = \frac{s}{s^2+a^2} \Rightarrow \mathcal{L}^{-1} \left\{ \frac{s}{s^2+a^2} \right\} = \cos at = g(t)$$

By convolution theorem: $\mathcal{L}^{-1} \{ \bar{f}(s) \bar{g}(s) \}$

$$= \int_0^t f(u) \cdot g(t-u) du$$

$$\mathcal{L}^{-1} \left\{ \frac{s^2}{(s^2+a^2)^2} \right\} = \int_0^t \cos au \cdot \cos a(t-u) du$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^t 2 \cos au \cdot \cos(at - au) du \\
&= \frac{1}{2} \int_0^t \cos(au + at - au) + \cos(au - at + au) du \\
&= \frac{1}{2} \int_0^t (\cos(at) + \cos(2au - at)) du \\
&= \frac{1}{2} \left[\int_0^t \cos at du + \int_0^t \cos(2au - at) du \right] \\
&= \frac{1}{2} \left[\cos at \cdot u + \frac{\sin(2au - at)}{2a} \right]_0^t \\
&= \frac{1}{2} \left[\cos at (t) + \frac{\sin(2at - at)}{2a} - \left(\cos at (0) + \frac{\sin(0 - at)}{2a} \right) \right] \\
&= \frac{1}{2} \left[t \cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a} \right] \\
&= \frac{1}{2} \left[t \cos at + \frac{\sin at}{a} \right] //
\end{aligned}$$

6) Solve the integral equation $y(t) = a \sin t - 2 \int_0^t y(u) \cos(t-u) du$

by convolution theorem:

By convolution theorem we have

If $\mathcal{L}^{-1}\{\bar{f}(s)\} = f(t)$ & $\mathcal{L}^{-1}\{\bar{g}(s)\} = g(t)$

Then $\mathcal{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} = f(t) * g(t)$

$$= \int_0^t f(u) g(t-u) du$$

(01)

$$L\{f(t) \times g(t)\} = \bar{f}(s) \bar{g}(s)$$

$$= L\{f(t) \otimes g(t)\}$$

$$\text{Given that } y(t) = a \sin t = 2 \int_0^t y(u) \cos(t-u) du$$

$$y(t) = a \sin t = 2 [y(t) \times \cos t] \quad \text{--- (1)}$$

Apply L.T on B.S of (1)

By convolution product

$$L\{y(t)\} = a L\{\sin t\} - 2 L\{y(t) \times \cos t\}$$

$$\Rightarrow L\{y(t)\} = a \frac{1}{s^2+1} - 2 L\{y(t)\} \cdot L\{\cos t\}$$

(By convolution Theorem).

$$\Rightarrow L\{y(t)\} = \frac{a}{s^2+1} - 2 L\{y(t)\} \frac{s}{s^2+1}$$

$$L\{y(t)\} \left(1 + \frac{2s}{s^2+1}\right) = \frac{a}{s^2+1}$$

$$\Rightarrow \left(\frac{s^2+1+2s}{s^2+1}\right) L\{y(t)\} = \frac{a}{s^2+1}$$

$$\Rightarrow L\{y(t)\} = \frac{a}{(s+1)^2}$$

$$\Rightarrow y(t) = L^{-1} \left\{ \frac{a}{(s+1)^2} \right\}$$

$$\Rightarrow y(t) = a e^{-t} L^{-1} \left\{ \frac{1}{s^2} \right\} \quad (\text{By F.S.T})$$

$$\Rightarrow y(t) = a e^{-t} t //$$

7) Solve the integral equation

$$f(t) = at + \int_0^t f(u) \sin(t-u) du, \quad t > 0 \text{ by convolution theorem.}$$

$$\text{Given that } f(t) = at + \int_0^t f(u) \sin(t-u) du$$

$$f(t) = at + [f(t) * \sin t] \quad \text{--- (1)}$$

Apply L.T on B.S of (1)

$$L\{f(t)\} = aL\{t\} + L\{f(t) * \sin t\}$$

$$\Rightarrow L\{f(t)\} = a \frac{1}{s^2} + L\{f(t)\} L\{\sin t\}$$

By convolution Theorem

$$\Rightarrow L\{f(t)\} = \frac{a}{s^2} + L\{f(t)\} \frac{1}{s^2+1}$$

$$L\{f(t)\} (s^2+1)$$

$$L\{f(t)\} \left(1 - \frac{1}{s^2+1}\right) = \frac{a}{s^2}$$

$$\Rightarrow L\left\{\left(\frac{s^2+1-1}{s^2+1}\right) L\{f(t)\}\right\} = \frac{a}{s^2}$$

$$L\{f(t)\} = \frac{a(s^2+1)}{(s^2)^2}$$

$$f(t) = L^{-1}\left\{\frac{a(s^2+1)}{(s^2)^2}\right\}$$

$$f(t) = aL^{-1}\left\{\frac{1}{s^2} + \frac{1}{s^4}\right\}$$

$$= a\left[L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{1}{s^4}\right\}\right] = a\left(t + \frac{t^3}{3!}\right)$$

$$f(t) = a \left(t + \frac{t^3}{3!} \right)$$

$$\Rightarrow f(t) = \frac{a}{6} (6t + 6^3)$$

$$8) \quad L^{-1} \left\{ \frac{s}{(s^2+a^2)^3} \right\} = L^{-1} \left\{ \frac{s}{(s^2+a^2)^2} \cdot \frac{1}{s^2+a^2} \right\}$$

$$\text{Let } \bar{f}(s) = \frac{s}{(s^2+a^2)^2} \text{ \& } \bar{g}(s) = \frac{1}{s^2+a^2}$$

$$L^{-1} \{ \bar{g}(s) \} = L^{-1} \left\{ \frac{1}{s^2+a^2} \right\}$$

$$= \frac{1}{a} \sin at$$

$$\text{Now } L^{-1} \{ \bar{f}(s) \} = L^{-1} \left\{ \frac{s}{(s^2+a^2)^2} \right\} = L^{-1} \left\{ \frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2} \right\}$$

$$\text{Let } \bar{p}(s) = \frac{s}{s^2+a^2} \Rightarrow L^{-1} \{ \bar{p}(s) \} = L^{-1} \left\{ \frac{s}{s^2+a^2} \right\} \\ = \cos at = p(t)$$

$$\& \quad \bar{q}(s) = \frac{1}{s^2+a^2} \Rightarrow L^{-1} \{ \bar{q}(s) \} = L^{-1} \left\{ \frac{1}{s^2+a^2} \right\} \\ = \frac{1}{a} \sin at = q(t)$$

$$\therefore L^{-1} \{ \bar{f}(s) \} = L^{-1} \left\{ \frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2} \right\}$$

$$= p(t) *_{\substack{t \\ t}} q(t) \quad (\text{By C.T})$$

$$= \int_0^t p(u) q(t-u) du$$

$$= \int_0^t \cos au \cdot \frac{1}{a} \sin(at-au) du$$

$$\begin{aligned}
&= \frac{1}{2a} \int_0^t 2 \cos au \sin(at - au) du \\
&= \frac{1}{2a} \int_0^t [\sin(au + at - au) - \sin(au - at + au)] du \\
&= \frac{1}{2a} \int_0^t [\sin at - \sin(2au - at)] du \\
&= \frac{1}{2a} \left[(\sin at) u + \frac{\cos(2au - at)}{2a} \right]_0^t \\
&= \frac{1}{2a} \left[t \sin at + \frac{1}{2a} \cos at - 0 - \frac{1}{2a} \cos(-at) \right] \\
&= \frac{1}{2a} \left[t \sin at + \frac{1}{2a} \cos at - \frac{1}{2a} \cos at \right] \\
&= \frac{1}{2a} t \sin at = f(t)
\end{aligned}$$

Again convolution Theorem

$$\begin{aligned}
L^{-1}\{ \bar{f}(s) \bar{g}(s) \} &= \int_0^t f(u) g(t-u) du \\
\Rightarrow L^{-1}\left\{ \frac{s}{(s^2+a^2)^2} \cdot \frac{1}{s^2+a^2} \right\} &= \int_0^t \frac{1}{2a} u \sin au \cdot \frac{1}{a} \sin a(t-u) du \\
&= \frac{1}{4a^2} \int_0^t u [2 \sin au \cdot \sin(at - au)] du \\
&= \frac{1}{4a^2} \int_0^t u [\cos(au - at + au) - \cos(at + at - au)] du \\
&= \frac{1}{4a^2} \left[\int_0^t u \cos(2au - at) du - \int_0^t u \cos at du \right]
\end{aligned}$$

$$= \frac{1}{4a^2} \left\{ \left[\frac{u \sin(2au - at)}{2a} + \frac{\cos(2au - at)}{(2a)^2} \right]_0^t - \left[\cos at \cdot \frac{u^2}{2} \right]_0^t \right\}$$

$$= \frac{1}{4a^2} \left\{ \left[\frac{t}{2a} \sin at + \frac{1}{4a^2} \cos at - 0 - \frac{\cos(-at)}{4a^2} \right] - \left(\frac{t^2}{2} \cos at \right) \right\}$$

$$= \frac{1}{4a^2} \left[\frac{t}{2a} \sin at + \frac{1}{4a^2} \cos at - \frac{1}{4a^2} \cos at - \frac{t^2}{2} \cos at \right]$$

$$= \frac{1}{8a^2} \left[\frac{1}{a} \sin at - t \cos at \right]$$

Solutions of ordinary differential equations by Laplace transform:

Consider a linear differential equations of order "n" with constant coefficient is

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \dots + a_{n-1} \frac{dy}{dt} + a_n y = f(t)$$

$$\Rightarrow y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_{n-1} y' + a_n y = f(t) \quad \text{--- (1)}$$

where $f(t)$ is a function of independent variable "t" and $y(0) = c_1, y'(0) = c_2, y''(0) = c_3$

$\dots y^{(n-1)}(0) = c_n$ (2) are the

Initial or boundary value conditions

Working Rule:

1) Taking L.T on both sides of ①

2) Apply $L\{y'(t)\} = s\bar{y}(s) - y(0)$

$$L\{y''(t)\} = s^2\bar{y}(s) - sy(0) - y'(0)$$

$$L\{y^{(n)}(t)\} = s^n\bar{y}(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \dots - y^{(n-1)}(0)$$

3) Substitute eq ② in the equation which is obtained in step ②

4) Transpose the term with " - " sign to the right.

5) Divide the coefficients of $\bar{y}$, getting $\bar{y}$ as a function of s

6) Resolve this equation of s into partial fractions.

7) Take the Inverse L.T of $\bar{y}$ obtained in step 5. This gives y as a function of t which is required solution.

1) Using L.T solve $(D^2 + 4D + 5)y = 5$, Given that $y(0) = 0, y'(0) = 0$

Given that $(D^2 + 4D + 5)y = 5$

$$\Rightarrow y'' + 4y' + 5y = 5 \quad \text{--- (1)}, \quad y(0) = 0, y'(0) = 0$$

Apply L.T on B.S of (1)

$$L\{y''\} + 4L\{y'\} + 5L\{y\} = L\{5\}$$

$$\Rightarrow s^2 L\{y(t)\} - sy(0) - y'(0) + 4[sL\{y(t)\} - y(0)] + 5L\{y(t)\} = \frac{5}{s}$$

$$\Rightarrow s^2 L\{y(t)\} - s(0) - 0 + 4sL\{y(t)\} - 4(0) + 5L\{y(t)\} = \frac{5}{s}$$

$$\Rightarrow L\{y(t)\}(s^2 + 4s + 5) = \frac{5}{s}$$

$$\Rightarrow L\{y(t)\} = \frac{5}{s(s^2 + 4s + 5)}$$

$$\Rightarrow y(t) = L^{-1}\left\{\frac{5}{s(s^2 + 4s + 5)}\right\}$$

$$= 5L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = 5 \int_0^t L^{-1}\{\bar{f}(s)\} dt$$

(By I.L.T of division by s')

$$\text{where } \bar{f}(s) = \frac{1}{s^2 + 4s + 5}$$

$$= 5 \int_0^t L^{-1}\left\{\frac{1}{s^2 + 4s + 5}\right\} dt$$

$$\begin{aligned}
&= 5 \int_0^t \mathcal{L}^{-1} \left\{ \frac{1}{(s+2)^2 + 1^2} \right\} dt \quad (\text{By I.L.T of F.S.T}) \\
&= 5 \int_0^t \mathcal{L}^{-1} \left\{ \frac{1}{(s+2)^2 + 1^2} \right\} dt \\
&= 5 \int_0^t e^{-2t} \sin t \, dt \\
&= 5 \left[\frac{e^{-2t}}{(-2)^2 + 1^2} (-2 \sin t - \cos t) \right]_0^t \\
&= 5 \left[\frac{e^{-2t}}{5} (-2 \sin t - \cos t) - \frac{1}{5} (0 - 1) \right] \\
&= 5 \left[\frac{1}{5} (1 - e^{-2t} (2 \sin t + \cos t)) \right] \\
&= 1 - e^{-2t} (2 \sin t + \cos t) //
\end{aligned}$$

2) Using L.T Method $(D^2 + 1)y = 6 \cos 2t$, if $y = 3$
 $dy = 1$, when $t = 0$

Given that
 D.E., $(D^2 + 1)y = 6 \cos 2t$

$$\Rightarrow y'' + y = 6 \cos 2t \quad \text{--- (1)}$$

Also given that,

$y = 3$, $dy = 1$ when $t = 0$ i.e.,

$$y(0) = 3, \quad y'(0) = 1$$

Apply L.T on eq (1), we get.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 6 \mathcal{L}\{\cos 2t\}$$

$$\Rightarrow s^2 \mathcal{L}\{y(t)\} - sy(0) - y'(0) + \mathcal{L}\{y(t)\} = \frac{6s}{s^2 + 4}$$

$$\Rightarrow s^2 L\{y(t)\} - s(3) - 1 + L\{y(t)\} = \frac{6s}{s^2+4}$$

$$\Rightarrow L\{y(t)\}(s^2+1) - 3s - 1 = \frac{6s}{s^2+4}$$

$$\Rightarrow L\{y(t)\}(s^2+1) = \frac{6s}{s^2+4} + 3s + 1$$

$$\Rightarrow L\{y(t)\} = \frac{6s}{(s^2+4)(s^2+1)} + \frac{3s}{s^2+1} + \frac{1}{s^2+1}$$

$$\Rightarrow L\{y(t)\}$$

$$\Rightarrow y(t) = 6 L^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\} + 3 L^{-1}\left\{\frac{s}{s^2+1}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\Rightarrow y(t) = 6 L^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\} + 3 \cos t + \sin t \quad \text{--- (2)}$$

From eq (2), consider

$$L^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\} = L^{-1}\left\{\frac{s}{(s^2+4)} \cdot \frac{1}{s^2+1}\right\}$$

$$\text{Let } \bar{f}(s) = \frac{s}{s^2+4} \Rightarrow L^{-1}\left\{\frac{s}{s^2+2^2}\right\} = \cos 2t = f(t)$$

$$\bar{g}(s) = \frac{1}{s^2+1} \Rightarrow L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = g(t)$$

By convolution Theorem,

$$L^{-1}\{\bar{f}(s)\bar{g}(s)\} = \int_0^t f(u)g(t-u)du$$

$$\Rightarrow L^{-1}\left\{\frac{s}{s^2+4} \cdot \frac{1}{s^2+1}\right\} = \int_0^t \cos 2u \cdot \sin(t-u)du$$

$$= \frac{1}{2} \int_0^t 2 \cos 2u \sin(t-u)du$$

$$= \frac{1}{2} \int_0^t [\sin(2u+t-u) - \sin(2u-t+u)]du$$

$$= \frac{1}{2} \int_0^t [\sin(u+t) - \sin(3u-t)] du$$

$$= \frac{1}{2} \left[\int_0^t \sin(u+t) du - \int_0^t \sin(3u-t) du \right] + \cos(3u-t)$$

$$= \frac{1}{2} \left[\left(-\cos(u+t) \right)_0^t - \left(-\frac{\cos(3u-t)}{3} \right)_0^t \right]$$

$$= \frac{1}{2} \left[-\cos 2t + \cos t + \frac{\cos 2t - \cos t}{3} \right]$$

$$\Rightarrow \frac{1}{2} \left[-\cos(2t) + \frac{\cos 2t}{3} - \left(-\cos t + \frac{\cos t}{3} \right) \right]$$

$$= \frac{1}{2} \left[\frac{-3\cos 2t + \cos 2t + 2\cos t}{3} \right]$$

$$= \frac{1}{2} \left[-\frac{2\cos 2t}{3} + \frac{2\cos t}{3} \right]$$

$$= -\frac{\cos 2t}{3} + \frac{1}{3} \cos t$$

sub eq (3) in (2)

$$y(t) = 6 \left[-\frac{\cos 2t}{3} + \frac{1}{3} \cos t \right] + 3\cos t + \sin t$$

$$= -2\cos 2t + 2\cos t + 3\cos t + \sin t$$

$$= 5\cos t - 2\cos 2t + \sin t$$

3-9-22

UNIT - IV

VECTOR CALCULUS

Vector differentiation

1) vector differential

operator $\cdot \nabla$ -del

(i) Gradient

(ii) Divergent

(iii) curl

2) Vector Identities

Vector Integration

1) Line integral

2) Surface integral

3) Volume integral

Vector Integral theorems

1) Green's theorem

2) Stoke's theorem

3) Divergence theorem

Scalar:- A physical quantity which has magnitude only is called as a scalar.

eg:- Every real no. is a scalar.

Vector:- A physical quantity which has both magnitude and direction is called vector.

eg:- Weight, displacement.

Scalar point function:- Let D be a domain of a function then for each variable $t \in D$ $\exists$ unique association of a scalar $f(t)$. then $f(t)$ is called as a scalar point function.

(or)

A scalar point function is a function that assigns a real no. (scalar) to each point of some region of space (domain).

If to each point (x, y, z) of a region R in space there is assigned a real no. 'u' $= \phi(x, y, z)$ then ϕ is called a scalar point function.

Vector point function:- Let D be a domain of a function then for each variable $t \in D$ if a unique association of vector $f(t)$ then $f(t)$ is called vector point function.

(or)

A vector point function is a function that assigns a vector to each point of some region of space, i.e. to each point (x, y, z) of a region R in space there is assigned a vector $\vec{F} = f(x, y, z)$ then $\vec{F}$ is called vector point function. Such function would have a representation $\vec{F} = f_1(x, y, z)\vec{i} + f_2(x, y, z)\vec{j} + f_3(x, y, z)\vec{k}$.

$$= f_1\vec{i} + f_2\vec{j} + f_3\vec{k}.$$

→ The vector differential operator is denoted by ∇ (read as del) and is defined as

$$\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$$

NOTE:-

→ Now we define the following quantities which involve the above operator.

- 1) Gradient of scalar point function
- 2) Divergent of a vector point function.
- 3) curl of a vector point function.

Gradient of a scalar point function:- Suppose $\phi(x, y, z)$ is a scalar point function then the gradient of a scalar point function is denoted with $\text{grad } \phi$ or $\nabla \phi$ and is defined as

$$\text{grad } \phi \text{ or } \nabla \phi = \left[\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right] \phi$$

$$= \left[\vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \right] = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

Eg:- Find $\nabla \phi$ if $\phi = x^2 y z$.

$$\nabla \phi = \left[\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right] (x^2 y z)$$

$$= \vec{i} \frac{\partial}{\partial x} (x^2 y z) + \vec{j} \frac{\partial}{\partial y} (x^2 y z) + \vec{k} \frac{\partial}{\partial z} (x^2 y z)$$

$$= \vec{i} (2x y z) + \vec{j} (x^2 z) + \vec{k} (x^2 y)$$

Nota:-

→ Gradient of a scalar point function is always a vector.

Properties:-

* If f & g are continuous and differentiable scalar point functions then

$$1) \text{grad}(f \pm g) = \text{grad } f \pm \text{grad } g$$

$$2) \text{grad}\left(\frac{f}{g}\right) = \frac{g \text{grad } f - f \text{grad } g}{g^2}$$

$$3) \text{grad}(fg) = g \text{grad } f + f \text{grad } g.$$

→ By the definition of gradient we can write

$$1) \text{grad}(f \pm g) = \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] (f \pm g)$$

$$= \left[\hat{i} \frac{\partial}{\partial x} (f \pm g) + \hat{j} \frac{\partial}{\partial y} (f \pm g) + \hat{k} \frac{\partial}{\partial z} (f \pm g) \right]$$

$$= \hat{i} \left[\frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x} \right] + \hat{j} \left[\frac{\partial f}{\partial y} \pm \frac{\partial g}{\partial y} \right] + \hat{k} \left[\frac{\partial f}{\partial z} \pm \frac{\partial g}{\partial z} \right]$$

$$= \left[\hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z} \right] \pm \left[\hat{i} \frac{\partial g}{\partial x} + \hat{j} \frac{\partial g}{\partial y} + \hat{k} \frac{\partial g}{\partial z} \right]$$

$$= \text{grad } f \pm \text{grad } g$$

(or)

$$\text{grad}(f \pm g) = \hat{x} \hat{i} \frac{\partial}{\partial x} (f \pm g) = \hat{x} \hat{i} \left(\frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x} \right)$$

$$= \hat{x} \hat{i} \frac{\partial f}{\partial x} \pm \hat{x} \hat{i} \frac{\partial g}{\partial x}$$

$$= \text{grad } f \pm \text{grad } g.$$

$$\therefore \text{grad}(f \pm g) = \text{grad } f \pm \text{grad } g.$$

$$2) \text{grad}\left(\frac{f}{g}\right) = \frac{g \text{grad } f - f \text{grad } g}{g^2}$$

$$\text{grad}\left(\frac{f}{g}\right) = \hat{x} \hat{i} \frac{\partial}{\partial x} \left(\frac{f}{g} \right)$$

$$= \hat{x} \hat{i} \left[\frac{g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x}}{g^2} \right]$$

$$= \left[\frac{1}{g^2} \hat{x} \hat{i} \left[g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x} \right] \right]$$

$$= \frac{1}{g^2} \left[g \hat{x} \hat{i} \frac{\partial f}{\partial x} - f \hat{x} \hat{i} \frac{\partial g}{\partial x} \right]$$

$$= \frac{1}{g^2} [g \nabla f - f \nabla g]$$

$$= \frac{1}{g^2} [g \text{grad } f - f \text{grad } g].$$

$$3) \text{grad}(fg) = g \text{grad} f + f \text{grad} g$$

$$\text{grad}(fg) = \hat{x}i - \frac{\partial}{\partial x}(fg)$$

$$= \hat{x}i \left[f \frac{\partial}{\partial x}(g) + g \frac{\partial}{\partial x}(f) \right]$$

$$= f \hat{x}i \frac{\partial}{\partial x}(g) + g \hat{x}i \frac{\partial}{\partial x}(f)$$

$$\nabla(fg) = f \nabla g + g \nabla f$$

$$\text{If } r = |\vec{r}| \text{ where } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$1) \text{Show that } d\phi = \nabla \phi \cdot d\vec{r}$$

$$2) \text{Show that } \nabla r = \frac{\vec{r}}{r}$$

$$3) \text{Show that } \nabla f(r) = \frac{f'(r)}{r} \vec{r}$$

$$4) \text{Show that } \nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$$

$$5) \text{Show that } \nabla(\log r) = \frac{\vec{r}}{r^2}$$

$$6) \text{Show that } \nabla(r^n) = nr^{n-2} \vec{r}$$

$$7) \text{If } f = x^2 + y^2 + z^2 \text{ then find } \nabla f$$

$$8) \text{If } f = xyz \text{ then find } \nabla f$$

$$6. \text{ } r = |\vec{r}| \text{ where } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad \text{--- (1)}$$

$$r = |x\hat{i} + y\hat{j} + z\hat{k}|$$

$$= \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2 \quad \text{--- (2)}$$

$$\text{P.D. (2) w.r.t } x, y, z$$

$$2x \frac{\partial r^2}{\partial x} = 2x \Rightarrow \frac{\partial r^2}{\partial x} = \frac{x}{r}$$

$$||\vec{r}|| \Rightarrow \frac{\partial r^2}{\partial y} = \frac{y}{r}$$

$$\Rightarrow \frac{\partial r^2}{\partial z} = \frac{z}{r}$$

$$1) \text{Show that } d\phi = \nabla \phi \cdot d\vec{r}$$

$$\text{Let } \phi(x, y, z) \text{ is a scalar point function.}$$

$$\nabla \phi = \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}$$

$$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\text{Now, the total derivative of scalar point function } \phi(x, y, z) \text{ is}$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \text{ is a scalar.}$$

$$= \left(\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$\therefore \text{The scalar can be written as dot product}$$

$$\therefore d\phi = \nabla\phi \cdot d\vec{r}.$$

8) Show that $\nabla r = \frac{\vec{r}}{r}$.

$$\begin{aligned} \nabla r &= \sum_i \hat{e}_i \frac{\partial r}{\partial x_i} \\ &= \sum_i \hat{e}_i \frac{x_i}{r} \\ &= \frac{1}{r} \sum_i x_i \hat{e}_i \\ &= \frac{1}{r} (x_1 \hat{e}_1 + y_1 \hat{e}_2 + z_1 \hat{e}_3) \\ &= \frac{\vec{r}}{r}. \end{aligned} \quad \left| \begin{aligned} \nabla r &= \left(\hat{e}_1 \frac{\partial}{\partial x} + \hat{e}_2 \frac{\partial}{\partial y} + \hat{e}_3 \frac{\partial}{\partial z} \right) \\ &= \hat{e}_1 \left(\frac{1}{r} (x_1^2 + y_1^2 + z_1^2)^{-1/2} \right) \cdot x_1 + \\ &\quad \hat{e}_2 \left(\frac{1}{r} (x_1^2 + y_1^2 + z_1^2)^{-1/2} \right) \cdot y_1 + \\ &\quad \hat{e}_3 \left(\frac{1}{r} (x_1^2 + y_1^2 + z_1^2)^{-1/2} \right) \cdot z_1 \\ &= \frac{1}{(x_1^2 + y_1^2 + z_1^2)^{1/2}} (x_1 \hat{e}_1 + y_1 \hat{e}_2 + z_1 \hat{e}_3) \\ &= \frac{1}{r} \cdot \vec{r}. \end{aligned} \right.$$

3) Show that $\nabla f(r) = \frac{f'(r)}{r} \vec{r}$.

$$\begin{aligned} \nabla(f(r)) &= \sum_i \hat{e}_i \frac{\partial}{\partial x_i} (f(r)) \\ &= \sum_i \hat{e}_i f'(r) \frac{\partial r}{\partial x_i} \\ &= \sum_i \hat{e}_i f'(r) \frac{x_i}{r} \\ &= f'(r) \frac{1}{r} \sum_i x_i \hat{e}_i \\ &= \frac{f'(r)}{r} \vec{r}. \end{aligned}$$

4) Show that $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^2}$

$$\begin{aligned} \nabla\left(\frac{1}{r}\right) &= \sum_i \hat{e}_i \frac{\partial}{\partial x_i} \left(\frac{1}{r}\right) \\ &= \sum_i \hat{e}_i \left(-\frac{1}{r^2}\right) \frac{\partial r}{\partial x_i} \\ &= \sum_i \hat{e}_i \left(-\frac{1}{r^2}\right) \frac{x_i}{r} \\ &= -\frac{1}{r^3} \sum_i x_i \hat{e}_i \\ &= -\frac{\vec{r}}{r^3} \end{aligned}$$

5) Show that $\nabla(\log r) = \frac{\vec{r}}{r^2}$

$$\begin{aligned} \nabla(\log r) &= \sum_i \hat{e}_i \frac{\partial}{\partial x_i} (\log r) \\ &= \sum_i \hat{e}_i \frac{1}{r} \cdot \frac{\partial r}{\partial x_i} \\ &= \frac{1}{r^2} \sum_i x_i \hat{e}_i \\ &= \frac{\vec{r}}{r^2} \end{aligned}$$

6) Show that $\nabla(r^n) = nr^{n-2} \vec{r}$

$$\begin{aligned}\nabla(r^n) &= \sum \vec{i} \frac{\partial}{\partial x} (r^n) \\ &= \sum \vec{i} \cdot nr^{n-1} \cdot \frac{\partial r}{\partial x} \\ &= \sum \vec{i} \cdot nr^{n-1} \cdot \frac{x}{r} \\ &= nr^{n-2} \sum x \vec{i} \\ &= nr^{n-2} \vec{r}.\end{aligned}$$

7) If $f = x^2 + y^2 + z^2$ then find ∇f .

$$\begin{aligned}\nabla f &= \sum \vec{i} \left(\frac{\partial f}{\partial x} \right) \\ &= \sum \vec{i} \left(2x \cdot \frac{\partial}{\partial x} \right)\end{aligned}$$

$$\text{grad } f = \nabla (x^2 + y^2 + z^2)$$

$$= \vec{i} \frac{\partial}{\partial x} (x^2 + y^2 + z^2) + \vec{j} \frac{\partial}{\partial y} (x^2 + y^2 + z^2) +$$

$$+ \vec{k} \frac{\partial}{\partial z} (x^2 + y^2 + z^2)$$

$$= \vec{i} (2x) + \vec{j} (2y) + \vec{k} (2z)$$

$$= 2\vec{r}$$

8) If $f = xyz$ then find ∇f .

$$\begin{aligned}\nabla f &= \sum \vec{i} \left(\frac{\partial f}{\partial x} \right) \\ &= \sum \vec{i} (yz) \\ &= \nabla (xyz) \\ &= \vec{i} \frac{\partial}{\partial x} (xyz) + \vec{j} \frac{\partial}{\partial y} (xyz) + \vec{k} \frac{\partial}{\partial z} (xyz) \\ &= yz \vec{i} + xz \vec{j} + xy \vec{k}\end{aligned}$$

6/9/22

Divergent of a vector point function:-

Suppose $\vec{F}$ is a continuously differentiable vector point function then divergent of a vector point function is denoted with $\text{div } \vec{F}$ or $\nabla \cdot \vec{F}$ and is defined as

$$\text{div } \vec{F} = \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} + \vec{j} \cdot \frac{\partial \vec{F}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{F}}{\partial z}$$

$$= \sum \vec{i} \cdot \frac{\partial \vec{F}}{\partial x}$$

Note:- If $\vec{F} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$ then $\text{div } \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Solenoidal vector:- A vector point function $\vec{F}$ is said to be solenoidal vector if $\text{div } \vec{F} = 0$.

Note:- Divergent of a vector point function is always a scalar.

1) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ find $\text{div } \vec{r}$.

$$\text{div } \vec{r} = \sum \vec{i} \cdot \frac{\partial \vec{r}}{\partial x} = \sum \vec{i} \cdot \vec{i} = \sum 1 = 1+1+1 = 3. \quad (\text{or})$$

$$\text{div } \vec{r} = \vec{i} \cdot \frac{\partial \vec{r}}{\partial x} + \vec{j} \cdot \frac{\partial \vec{r}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{r}}{\partial z}$$

$$= \vec{i} \cdot \vec{i} + \vec{j} \cdot \vec{j} + \vec{k} \cdot \vec{k} = 1+1+1$$

$$= 3$$

$$(or) \text{div } \vec{r} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x} (x) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z)$$

$$= 1+1+1 = 3.$$

2) If $f = x^2 y \vec{i} - 2xz \vec{j} + 2yz \vec{k}$ find $\text{div } f$.

$$G.T \quad f = x^2 y \vec{i} - 2xz \vec{j} + 2yz \vec{k}.$$

$$\text{div } f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x} (x^2 y) + \frac{\partial}{\partial y} (-2xz) + \frac{\partial}{\partial z} (2yz)$$

$$= 2xy - 0 + 2y$$

$$= 2y(x+1).$$

3) Find $\text{div } f$ when $f = \text{grad } (x^3 + y^3 + z^3 - 3xyz)$.

$$G.T \quad f = \text{grad } (x^3 + y^3 + z^3 - 3xyz)$$

$$f = \left[\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right] (x^3 + y^3 + z^3 - 3xyz)$$

$$= \vec{i} \frac{\partial}{\partial x} (x^3 + y^3 + z^3 - 3xyz) + \vec{j} \frac{\partial}{\partial y} (x^3 + y^3 + z^3 - 3xyz)$$

$$+ \vec{k} \frac{\partial}{\partial z} (x^3 + y^3 + z^3 - 3xyz)$$

$$f = \vec{i} (3x^2 - 3yz) + \vec{j} (3y^2 - 3xz) + \vec{k} (3z^2 - 3xy)$$

$$\text{div } \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x} (3x^2 - 3yz) + \frac{\partial}{\partial y} (3y^2 - 3xz) + \frac{\partial}{\partial z} (3z^2 - 3xy)$$

$$= 6x + 6y + 6z.$$

$$= 6(x+y+z)$$

4) Show that the vector $\vec{f} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is solenoidal.

G.T. $\vec{f} = yz\vec{i} + zx\vec{j} + xy\vec{k}$. To prove that $\vec{f}$ is solenoidal it is enough to prove that $\text{div } \vec{f} = 0$.

$$\text{div } \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x} (yz) + \frac{\partial}{\partial y} (zx) + \frac{\partial}{\partial z} (xy)$$

$$= 0 + 0 + 0.$$

$\therefore \text{div } \vec{f} = 0$, $\therefore \vec{f}$ is solenoidal.

5) Determine the constant a so that the vector

$\vec{f} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x-az)\vec{k}$ is solenoidal.

G.T. $\vec{f} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x-az)\vec{k}$ is

solenoidal. $\Rightarrow \text{div } \vec{f} = 0$.

$$\operatorname{div} \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 0$$

$$\Rightarrow \frac{\partial}{\partial x} (x+3y) + \frac{\partial}{\partial y} (y-2x) + \frac{\partial}{\partial z} (x-ax) = 0$$

$$\Rightarrow 1 + 1 - a = 0$$

$$2 - a = 0$$

$$\boxed{a=2}$$

6) Show that $\vec{r}/r^3$ is irrotational

$$\text{div}\left(\frac{\vec{r}}{r^3}\right) = \vec{i} \cdot \frac{\partial}{\partial x} \left[\frac{\vec{r}}{r^3} \right]$$

$$= \vec{i} \cdot \frac{\partial}{\partial x} \left[\vec{r} \cdot \frac{1}{r^3} \right]$$

$$= \vec{i} \cdot \left[\frac{\partial \vec{r}}{\partial x} \cdot \frac{1}{r^3} + \vec{r} \left(\frac{-3}{r^4} \right) \frac{\partial r}{\partial x} \right]$$

$$= \vec{i} \cdot \left[\frac{\vec{i}}{r^3} - \frac{3\vec{r}}{r^4} \frac{x}{r} \right]$$

$$= \frac{1}{r^3} (\vec{i} \cdot \vec{i}) - \frac{3}{r^5} (\vec{i} \cdot \vec{r}) x$$

$$= \frac{1}{r^3} - \frac{3}{r^5} x^2 = \frac{3}{r^3} - \frac{3}{r^5} (r^2)$$

$$= \frac{3}{r^3} - \frac{3}{r^3} = 0.$$

7) Prove that $\text{div}\left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\text{div}\left(\frac{\vec{r}}{r}\right) = \vec{i} \cdot \frac{\partial}{\partial x} \left(\vec{r} \cdot \frac{1}{r} \right)$$

$$= \vec{i} \cdot \left(\vec{r} \cdot \frac{\partial}{\partial x} \frac{1}{r} + \frac{1}{r} \frac{\partial \vec{r}}{\partial x} \right)$$

$$= \vec{i} \cdot \left[\frac{\partial \vec{r}}{\partial x} \cdot \frac{1}{r} + \vec{r} \left(\frac{-1}{r^2} \right) \frac{\partial r}{\partial x} \right]$$

$$= \vec{i} \cdot \left[\frac{\vec{i}}{r} - \frac{\vec{r}}{r^2} \frac{x}{r} \right]$$

$$= \leq \frac{1}{r} (\vec{i} \cdot \vec{i}) - \leq \frac{x}{r^3} (\vec{i} \cdot \vec{r})$$

$$= \leq \frac{1}{r} - \leq \frac{x}{r^3} (x) = \frac{3}{r} - \frac{1}{r^3} \leq x^2$$

$$= \frac{3}{r} - \frac{r^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$$\therefore \operatorname{div} \left(\frac{\vec{r}}{r} \right) = \frac{2}{r}$$

Curl of a vector point function:

Suppose $\vec{F}$ is a continuously differentiable vector point function the curl of a vector point function is denoted with $\operatorname{curl} \vec{F}$ or $\nabla \times \vec{F}$ is defined as

$$\operatorname{curl} \vec{F} = \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}$$

$$= \vec{i} \times \frac{\partial \vec{F}}{\partial x}$$

Note:- If $\vec{F} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$ then

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Irrrotational vectors- A vector point function $\vec{F}$ is said to be irrotational if $\text{curl } \vec{F} = 0$.

Note- If $\vec{F}$ is irrotational then there will always exist a scalar $\phi(x, y, z)$ such that $\vec{F} = \text{grad } \phi$ where ϕ is called the scalar potential.

1) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ find $\text{curl } \vec{r}$.

$$\begin{aligned}\text{curl } \vec{r} &= \vec{i} \times \frac{\partial \vec{r}}{\partial x} + \vec{j} \times \frac{\partial \vec{r}}{\partial y} + \vec{k} \times \frac{\partial \vec{r}}{\partial z} \\ &= \vec{i} \times \vec{i} + \vec{j} \times \vec{j} + \vec{k} \times \vec{k} \\ &= 0.\end{aligned}$$

$$\text{curl } \vec{r} = \sum \vec{i} \times \frac{\partial \vec{r}}{\partial x} = \sum (\vec{i} \times \vec{i}) = 0.$$

2) Find $\text{curl } \vec{f}$ when $\vec{f} = \text{grad} (x^3 + y^3 + z^3 - 3xyz)$

$$\begin{aligned}\text{grad} (x^3 + y^3 + z^3 - 3xyz) &= \vec{i} \frac{\partial}{\partial x} (x^3 + y^3 + z^3 - 3xyz) \\ &\quad + \vec{j} \frac{\partial}{\partial y} (x^3 + y^3 + z^3 - 3xyz) + \vec{k} \frac{\partial}{\partial z} (x^3 + y^3 + z^3 - 3xyz)\end{aligned}$$

$$\text{grad} (x^3 + y^3 + z^3 - 3xyz) = \vec{i} (3x^2 - 3yz) + \vec{j} (3y^2 - 3xz) + \vec{k} (3z^2 - 3xy).$$

$$\text{curl } \vec{f} =$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$\vec{i} \left(\frac{\partial}{\partial y} (3z^2 - 3xy) - \frac{\partial}{\partial z} (3y^2 - 3xz) \right) -$$

$$\vec{j} \left(\frac{\partial}{\partial x} (3z^2 - 3xy) - \frac{\partial}{\partial z} (3x^2 - 3yz) \right) +$$

$$\vec{k} \left(\frac{\partial}{\partial x} (3y^2 - 3xz) - \frac{\partial}{\partial y} (3x^2 - 3yz) \right)$$

$$\begin{aligned}&= (\vec{i} [-3x + 3z] - \vec{j} [-3y + 3y] + \vec{k} [-3z + 3z]) \\ &= \vec{i} (0) + \vec{j} (0) - \vec{j} (0) = 0.\end{aligned}$$

3) If $\vec{f} = x^2y\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ find $\text{curl } \vec{f}$ at the point $(-1, -1, 1)$.

$$\vec{f} = x^2y\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}.$$

$$\text{curl } \vec{f} =$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$\bar{i} \left[\frac{\partial}{\partial y} (-3yx^2) - \frac{\partial}{\partial z} (2x^2yz) \right] -$$

$$\bar{j} \left[\frac{\partial}{\partial x} (-3yx^2) - \frac{\partial}{\partial z} (x^2y) \right] +$$

$$\bar{k} \left[\frac{\partial}{\partial x} (2x^2yz) - \frac{\partial}{\partial y} (x^2y) \right]$$

$$= \bar{i} [-3x^2 - 2x^2y] \cdot \bar{j} [-0 - 0] + \bar{k} [4xy - x^2]$$

$$= \bar{i} (-3x^2 - 2x^2y) + \bar{k} (4xy - x^2)$$

At point (1, -1, 1)

$$\bar{i} (-3(1) - 2(1)(-1)) + \bar{k} (4(1)(-1) - (-1)^2)$$

$$\bar{i} (-3 + 2) + \bar{k} (-4 - 1)$$

$$= -\bar{i} - 5\bar{k}$$

4) show that $\vec{f} = (\sin y + z)\bar{i} + (x \cos y - z)\bar{j} +$

$(x - y)\bar{k}$ is irrotational.

$$\vec{f} = (\sin y + z)\bar{i} + (x \cos y - z)\bar{j} + (x - y)\bar{k}$$

To prove that $\vec{f}$ is irrotational it is enough to prove that $\text{curl } \vec{f} = 0$

$$\text{curl } \vec{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & x \cos y - z & x - y \end{vmatrix}$$

$$= \bar{i} \left[\frac{\partial}{\partial y} (x - y) - \frac{\partial}{\partial z} (x \cos y - z) \right] - \bar{j} \left[\frac{\partial}{\partial x} (x - y) - \right.$$

$$\left. \frac{\partial}{\partial z} (\sin y + z) \right] + \bar{k} \left[\frac{\partial}{\partial x} (x \cos y - z) - \frac{\partial}{\partial y} (\sin y + z) \right]$$

$$= \bar{i} [(0 - 1) - (0 - 1)] - \bar{j} [(1 - 0) - (0 + 1)] + \bar{k} [\cos y - \cos y]$$

$$= \bar{i} (-1 + 1) - \bar{j} (1 - 1) + \bar{k} (0)$$

$$= 0\bar{i} + 0\bar{j} + 0\bar{k} = 0$$

Hence proved.

$\therefore \vec{f}$ is irrotational.

5) Find the constants a, b, c so that the vector

$\vec{f} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

Given the vector is irrotational i.e. $\text{curl } \vec{f} = 0$

$\vec{f}$ is of the form $f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$

$$\therefore \text{curl } \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \vec{i} \left[\frac{\partial}{\partial y} (4x+cy+2z) - \frac{\partial}{\partial z} (bx-3y-z) \right] - \vec{j} \left[\frac{\partial}{\partial x} (4x+cy+2z) - \frac{\partial}{\partial z} (x+2y+az) \right] + \vec{k} \left[\frac{\partial}{\partial x} (bx-3y-z) - \frac{\partial}{\partial y} (x+2y+az) \right]$$

$$\Rightarrow \vec{i} (c+1) - \vec{j} (4-a) + \vec{k} (b-2) = 0$$

$$c+1=0 \Rightarrow c=-1$$

$$-4+a=0 \Rightarrow a=4$$

$$b-2=0 \Rightarrow b=2$$

6) Prove that $\nabla \cdot (\vec{a} \times \vec{f}) = -\vec{a} \cdot \text{curl } \vec{f}$

where $\vec{a}$ is a constant vector.

$$\nabla \cdot (\vec{a} \times \vec{f}) = \sum \vec{i} \cdot \frac{\partial}{\partial x} (\vec{a} \times \vec{f})$$

$$= \sum \vec{i} \left[\frac{\partial a}{\partial x} \times \vec{f} + \vec{a} \times \frac{\partial \vec{f}}{\partial x} \right]$$

$$\because \frac{\partial a}{\partial x} (\vec{a} \times \vec{f}) = \frac{\partial a}{\partial x} \times \vec{f} + \vec{a} \times \frac{\partial \vec{f}}{\partial x}$$

$$= \sum \vec{i} \left[0 + \vec{a} \times \frac{\partial \vec{f}}{\partial x} \right]$$

$$= \sum \vec{i} \left(\vec{a} \times \frac{\partial \vec{f}}{\partial x} \right) \quad \text{--- (1)}$$

(R.H.S.)

$$\text{curl } \vec{f} = \sum \vec{i} \times \frac{\partial \vec{f}}{\partial x} \quad \cdot [\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}]$$

$$= \sum \vec{i} \left[\frac{\partial}{\partial x} (\vec{a} \times \vec{f}) \right]$$

$$= \sum \vec{i} \left[0 + \vec{a} \times \frac{\partial \vec{f}}{\partial x} \right]$$

$$= \sum \vec{i} \left(\vec{a} \times \frac{\partial \vec{f}}{\partial x} \right)$$

$$= \sum (\vec{i} \times \vec{a}) \cdot \frac{\partial \vec{f}}{\partial x}$$

$$= \sum -(\vec{a} \times \vec{i}) \cdot \frac{\partial \vec{f}}{\partial x} = -\sum \vec{a} \cdot (\vec{i} \times \frac{\partial \vec{f}}{\partial x})$$

$$= -\vec{a} \cdot \left(\sum \vec{i} \times \frac{\partial \vec{f}}{\partial x} \right)$$

$$= -\vec{a} \cdot \text{curl } \vec{f}$$

7) Prove that $\text{curl}(\bar{a} \times \bar{r}) = 2\bar{a}$ where $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ is a constant vector.

$$\text{curl}(\bar{a} \times \bar{r}) = \nabla \times \frac{\partial}{\partial x} (\bar{a} \times \bar{r})$$

$$= \nabla \times \left[\frac{\partial \bar{a}}{\partial x} \times \bar{r} + \bar{a} \times \frac{\partial \bar{r}}{\partial x} \right]$$

$$\therefore \frac{\partial}{\partial x} (\bar{a} \times \bar{r}) = \frac{\partial \bar{a}}{\partial x} \times \bar{r} + \bar{a} \times \frac{\partial \bar{r}}{\partial x}$$

$$= \nabla \times [0 + \bar{a} \times \bar{r}]$$

$$= \nabla \times (\bar{a} \times \bar{r})$$

$$= \nabla \times [(\bar{i} \cdot \bar{i})\bar{a} - (\bar{i} \cdot \bar{a})\bar{i}] \quad [\because \bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}]$$

$$= \nabla \times [\bar{a} - a_1\bar{i}]$$

$$= \nabla \times \bar{a} - \nabla \times a_1\bar{i}$$

$$= 3\bar{a} - (a_1\bar{i} + a_2\bar{j} + a_3\bar{k})$$

$$= 3\bar{a} - \bar{a}$$

$$= 2\bar{a}$$

8. Prove that $\text{curl} \left(\frac{\bar{a} \times \bar{r}}{r^3} \right) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5} (\bar{a} \cdot \bar{r})$

$$\text{curl} \left(\frac{\bar{a} \times \bar{r}}{r^3} \right) = \nabla \times \frac{\partial}{\partial x} \left(\frac{\bar{a} \times \bar{r}}{r^3} \right) \quad \text{--- (1)}$$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \quad , \quad \frac{\partial \bar{r}}{\partial x} = \bar{i}, \quad \frac{\partial \bar{r}}{\partial y} = \bar{j}, \quad \frac{\partial \bar{r}}{\partial z} = \bar{k}$$

$$r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2} \quad , \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

from (1) consider,

$$\frac{\partial}{\partial x} \left(\frac{\bar{a} \times \bar{r}}{r^3} \right) = \bar{a} \times \frac{\partial}{\partial x} \left(\frac{\bar{r}}{r^3} \right)$$

$$= \bar{a} \times \frac{\partial}{\partial x} \left(\bar{r} \frac{1}{r^3} \right)$$

$$= \bar{a} \times \left[\frac{\partial \bar{r}}{\partial x} \frac{1}{r^3} + \bar{r} \left(\frac{-3}{r^4} \right) \frac{\partial r}{\partial x} \right]$$

$$= \bar{a} \times \left[\frac{\bar{i}}{r^3} - \frac{3\bar{r}}{r^4} \frac{x}{r} \right]$$

$$= \bar{a} \times \left[\frac{\bar{i}}{r^3} - \frac{3x\bar{r}}{r^5} \right]$$

$$= \frac{1}{r^3} (\bar{a} \times \bar{r}) - \frac{3x}{r^5} (\bar{a} \times \bar{r}) \quad \text{--- (2)}$$

Again from (1)

$$\text{Consider } \bar{r} \times \frac{\partial}{\partial x} \left(\frac{\bar{a} \times \bar{r}}{r^3} \right) = \bar{r} \times \left[\frac{(\bar{a} \times \bar{r})}{r^3} - \frac{3x}{r^5} (\bar{a} \times \bar{r}) \right]$$

$$= \frac{1}{r^3} \left\{ \bar{r} \times (\bar{a} \times \bar{r}) - \frac{3x}{r^5} (\bar{r} \times (\bar{a} \times \bar{r})) \right\}$$

$$= \frac{1}{r^3} \left[(\bar{r} \cdot \bar{r})\bar{a} - (\bar{r} \cdot \bar{a})\bar{r} \right] - \frac{3x}{r^5} \left[(\bar{r} \cdot \bar{r})\bar{a} - (\bar{r} \cdot \bar{a})\bar{r} \right]$$

$$= \frac{1}{r^3} (\bar{a} - a_1\bar{i}) - \frac{3x}{r^5} (x\bar{a} - a_1\bar{r}) \quad \text{--- (3)}$$

$\therefore$ (1) can be written as

$$\begin{aligned}
 \text{curl} \left(\frac{\bar{a} \times \bar{r}}{r^3} \right) &= \leq \bar{i} \times \frac{\partial}{\partial x} \left(\frac{\bar{a} \times \bar{r}}{r^3} \right) \\
 &= \leq \left[\frac{1}{r^3} (\bar{a} - a_1 \bar{i}) - \frac{3x}{r^5} (x\bar{a} - a_1 \bar{i}) \right] \\
 &= \frac{1}{r^5} \left[\leq \bar{a} - \leq a_1 \bar{i} \right] - \frac{3x}{r^5} \leq x^2 + \frac{5x}{r^5} \leq a_1 \bar{i} \\
 &= \frac{1}{r^3} \cdot (3\bar{a} - \bar{a}) - \frac{3\bar{a}}{r^5} (x^2) + \frac{3x}{r^5} [a_1 x + a_2 y + a_3 z] \\
 &= \frac{2\bar{a}}{r^3} - \frac{3\bar{a}}{r^3} + \frac{3x}{r^5} [a_1 \bar{i} + a_2 \bar{j} + a_3 \bar{k}] \cdot (x\bar{i} + y\bar{j} + z\bar{k}) \\
 &= \frac{-\bar{a}}{r^3} + \frac{3x}{r^5} (\bar{a} \cdot \bar{r}) \quad \text{//}
 \end{aligned}$$

8-9-22

Operators:-

1) Scalar differential operator:- The operator

$$\begin{aligned}
 (\bar{a} \cdot \nabla) &= \bar{a} \cdot \bar{i} \frac{\partial}{\partial x} + \bar{a} \cdot \bar{j} \frac{\partial}{\partial y} + \bar{a} \cdot \bar{k} \frac{\partial}{\partial z} \\
 (\bar{a} \cdot \nabla) \phi &= \bar{a} \cdot \bar{i} \frac{\partial \phi}{\partial x} + \bar{a} \cdot \bar{j} \frac{\partial \phi}{\partial y} + \bar{a} \cdot \bar{k} \frac{\partial \phi}{\partial z} \\
 &= (\bar{a} \cdot \bar{i}) \frac{\partial \phi}{\partial x} + (\bar{a} \cdot \bar{j}) \frac{\partial \phi}{\partial y} + (\bar{a} \cdot \bar{k}) \frac{\partial \phi}{\partial z}
 \end{aligned}$$

2) vector differential operator:-

$$\begin{aligned}
 (\bar{a} \times \nabla) &= (\bar{a} \times \bar{i}) \frac{\partial}{\partial x} + (\bar{a} \times \bar{j}) \frac{\partial}{\partial y} + (\bar{a} \times \bar{k}) \frac{\partial}{\partial z} \\
 (\bar{a} \times \nabla) \phi &= (\bar{a} \times \bar{i}) \frac{\partial \phi}{\partial x} + (\bar{a} \times \bar{j}) \frac{\partial \phi}{\partial y} + (\bar{a} \times \bar{k}) \frac{\partial \phi}{\partial z} \\
 (\bar{a} \times \nabla) \cdot \bar{f} &= (\bar{a} \times \bar{i}) \cdot \frac{\partial \bar{f}}{\partial x} + (\bar{a} \times \bar{j}) \cdot \frac{\partial \bar{f}}{\partial y} + (\bar{a} \times \bar{k}) \cdot \frac{\partial \bar{f}}{\partial z} \\
 (\bar{a} \times \nabla) \times \bar{f} &= (\bar{a} \times \bar{i}) \times \frac{\partial \bar{f}}{\partial x} + (\bar{a} \times \bar{j}) \times \frac{\partial \bar{f}}{\partial y} + (\bar{a} \times \bar{k}) \times \frac{\partial \bar{f}}{\partial z}
 \end{aligned}$$

Laplacian Operator:-

Consider

$$\nabla \cdot \nabla \phi = \text{div}(\text{grad } \phi)$$

$$= \leq \bar{i} \cdot \frac{\partial}{\partial x} (\nabla \phi)$$

$$= \leq \bar{i} \cdot \frac{\partial}{\partial x} \left[\bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z} \right]$$

$$= \leq \bar{i} \cdot \left[\bar{i} \frac{\partial^2 \phi}{\partial x^2} + \bar{j} \frac{\partial^2 \phi}{\partial x \partial y} + \bar{k} \frac{\partial^2 \phi}{\partial x \partial z} \right]$$

$$= \leq \frac{\partial^2 \phi}{\partial x^2}$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \phi$$

$$= \nabla^2 \phi$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a Laplacian operator.

Q) Show that $\nabla^2\left(\frac{1}{r}\right) = 0$.

$$\nabla^2 \phi = \nabla \cdot \nabla \phi$$

$$\nabla^2\left(\frac{1}{r}\right) = \nabla \cdot \nabla\left(\frac{1}{r}\right) \quad \text{--- (1)}$$

Consider,

$$\nabla\left(\frac{1}{r}\right) = \sum \bar{i} \frac{\partial}{\partial x} \left(\frac{1}{r}\right)$$

$$= \sum \bar{i} \left(-\frac{1}{r^2}\right) \frac{\partial r}{\partial x}$$

$$= \sum \bar{i} \frac{-1}{r^2} \frac{\partial r}{\partial x}$$

$$= \frac{-1}{r^3} \sum \bar{i} x_i = \frac{-\vec{r}}{r^3}$$

$$\therefore \nabla \cdot \nabla\left(\frac{1}{r}\right) = \sum \bar{i} \cdot \frac{\partial}{\partial x} \left(\frac{-\vec{r}}{r^3}\right)$$

$$= -\sum \bar{i} \cdot \left[\frac{\partial}{\partial x} \frac{-1}{r^3} + \bar{r} \left(\frac{-3}{r^4} \right) \frac{\partial r}{\partial x} \right]$$

$$= -\sum \bar{i} \cdot \left[\frac{1}{r^3} - \frac{3\bar{r}}{r^4} \cdot \frac{x}{r} \right]$$

$$= -\sum \left(\frac{\bar{i} \cdot \bar{i}}{r^3} + \sum \frac{3x}{r^5} (\bar{i} \cdot \bar{r}) \right)$$

$$= \frac{-3}{r^3} + \frac{3x}{r^5} \sum (\bar{i} \cdot \bar{r}) x$$

$$= \frac{-3}{r^3} + \left(\frac{-3}{r^5} \right) \sum x^2$$

$$= \frac{-3}{r^3} + \frac{3}{r^5} (r^2) = \frac{-3}{r^3} + \frac{3}{r^3} = 0$$

Vector Identities:-

If $\vec{a}$ and $\vec{b}$ are differentiable functions and ϕ is a differentiable scalar then prove that

$$1) \operatorname{div}(\phi \vec{a}) = \operatorname{grad}(\phi) \cdot \vec{a} + \phi \operatorname{div} \vec{a}$$

$$2) \operatorname{curl}(\phi \vec{a}) = \operatorname{grad}(\phi) \times \vec{a} + \phi \operatorname{curl} \vec{a}$$

$$3) \operatorname{grad}(\vec{a} \cdot \vec{b}) = (\vec{b} \cdot \nabla) \vec{a} + (\vec{a} \cdot \nabla) \vec{b} + \vec{b} \times \operatorname{curl} \vec{a} + \vec{a} \times \operatorname{curl} \vec{b}$$

$$4) \operatorname{div}(\vec{a} \times \vec{b}) = \vec{b} \cdot \operatorname{curl} \vec{a} - \vec{a} \cdot \operatorname{curl} \vec{b}$$

$$5) \operatorname{curl}(\vec{a} \times \vec{b}) = \vec{a} \operatorname{div} \vec{b} - \vec{b} \operatorname{div} \vec{a} + (\vec{b} \cdot \nabla) \vec{a} - (\vec{a} \cdot \nabla) \vec{b}$$

$$6) \operatorname{curl}(\operatorname{grad} \phi) = 0$$

$$7) \operatorname{div}(\operatorname{curl} \vec{f}) = 0$$

8) If f and g are two scalar point functions.

$$\text{Then } \operatorname{div}(f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g$$

$$9) \operatorname{curl}(\operatorname{curl} \vec{a}) = \operatorname{grad}(\operatorname{div} \vec{a}) - \nabla^2 \vec{a}$$

$$① \operatorname{div}(\phi \bar{a}) = \operatorname{grad}(\phi) \cdot \bar{a} + \phi \operatorname{div} \bar{a}$$

$$\text{Consider } \operatorname{div}(\phi \bar{a}) = \sum \bar{i} \cdot \frac{\partial}{\partial x} (\phi \bar{a})$$

$$= \sum \bar{i} \cdot \left[\frac{\partial \phi}{\partial x} \bar{a} + \phi \frac{\partial \bar{a}}{\partial x} \right]$$

$$= \sum \bar{i} \cdot \left(\frac{\partial \phi}{\partial x} \bar{a} \right) + \sum \bar{i} \cdot \left(\phi \frac{\partial \bar{a}}{\partial x} \right)$$

$$= \left(\sum \bar{i} \frac{\partial \phi}{\partial x} \right) \cdot \bar{a} + \left(\sum \bar{i} \cdot \frac{\partial \bar{a}}{\partial x} \right) \phi$$

$$= \operatorname{grad} \phi \cdot \bar{a} + \operatorname{div} \bar{a} \phi$$

$$\therefore \operatorname{div}(\phi \bar{a}) = \operatorname{grad} \phi \cdot \bar{a} + \operatorname{div} \bar{a} \phi \quad (or)$$

$$\nabla \cdot (\phi \bar{a}) = \nabla \phi \cdot \bar{a} + \phi (\nabla \cdot \bar{a})$$

$$② \operatorname{curl}(\phi \bar{a}) = \operatorname{grad}(\phi) \times \bar{a} + \phi \operatorname{curl} \bar{a}$$

$$\text{Consider } \operatorname{curl}(\phi \bar{a}) = \sum \bar{i} \times \frac{\partial}{\partial x} (\phi \bar{a})$$

$$= \sum \bar{i} \times \left[\frac{\partial \phi}{\partial x} \bar{a} + \phi \frac{\partial \bar{a}}{\partial x} \right]$$

$$= \sum \bar{i} \times \left(\frac{\partial \phi}{\partial x} \bar{a} \right) + \sum \bar{i} \times \left(\phi \frac{\partial \bar{a}}{\partial x} \right)$$

$$= \left(\sum \bar{i} \frac{\partial \phi}{\partial x} \right) \times \bar{a} + \left(\sum \bar{i} \times \frac{\partial \bar{a}}{\partial x} \right) \phi$$

$$= \operatorname{grad} \phi \times \bar{a} + \operatorname{curl} \bar{a} \phi$$

$$\therefore \operatorname{curl}(\phi \bar{a}) = \operatorname{grad}(\phi) \times \bar{a} + \phi \operatorname{curl} \bar{a} \quad (or)$$

$$\nabla \times (\phi \bar{a}) = \nabla \phi \times \bar{a} + (\nabla \times \bar{a}) \phi$$

$$③ \operatorname{grad}(\bar{a} \cdot \bar{b}) = (\bar{b} \cdot \nabla) \bar{a} + (\bar{a} \cdot \nabla) \bar{b} + \bar{b} \times \operatorname{curl} \bar{a} + \bar{a} \times \operatorname{curl} \bar{b}$$

$$\nabla(\bar{a} \cdot \bar{b}) = (\bar{b} \cdot \nabla) \bar{a} + (\bar{a} \cdot \nabla) \bar{b} + \bar{b} \times (\nabla \times \bar{a}) + \bar{a} \times (\nabla \times \bar{b})$$

Consider,

$$\bar{b} \times \operatorname{curl} \bar{a} = \bar{b} \times \sum \bar{i} \times \frac{\partial \bar{a}}{\partial x}$$

$$= \sum (\bar{b} \times (\bar{i} \times \frac{\partial \bar{a}}{\partial x}))$$

$$= \sum \left[(\bar{b} \cdot \frac{\partial \bar{a}}{\partial x}) \bar{i} - (\bar{b} \cdot \bar{i}) \frac{\partial \bar{a}}{\partial x} \right]$$

$$= \sum (\bar{b} \cdot \frac{\partial \bar{a}}{\partial x}) \bar{i} - \sum (\bar{b} \cdot \bar{i}) \frac{\partial \bar{a}}{\partial x}$$

$$= \sum (\bar{b} \cdot \frac{\partial \bar{a}}{\partial x}) \bar{i} - \sum (\bar{b} \cdot \bar{i} \frac{\partial}{\partial x}) \bar{a}$$

$$= \sum (\bar{b} \cdot \frac{\partial \bar{a}}{\partial x}) \bar{i} - \bar{b} \cdot \left(\sum \bar{i} \frac{\partial}{\partial x} \right) \bar{a}$$

$$= \sum (\bar{b} \cdot \frac{\partial \bar{a}}{\partial x}) \bar{i} - (\bar{b} \cdot \nabla) \bar{a} \quad \text{--- (1)}$$

Similarly $\bar{a} \times \operatorname{curl} \bar{b}$

$$= \sum (\bar{a} \cdot \frac{\partial \bar{b}}{\partial x}) \bar{i} - (\bar{a} \cdot \nabla) \bar{b} \quad \text{--- (2)}$$

① + ②

$$\bar{b} \times \operatorname{curl} \bar{a} + \bar{a} \times \operatorname{curl} \bar{b} = \sum (\bar{b} \cdot \frac{\partial \bar{a}}{\partial x}) \bar{i} - (\bar{b} \cdot \nabla) \bar{a} + \sum (\bar{a} \cdot \frac{\partial \bar{b}}{\partial x}) \bar{i} - (\bar{a} \cdot \nabla) \bar{b}$$

$$\leq (\bar{a} \cdot \frac{\partial \bar{b}}{\partial x}) \bar{i} - (\bar{a} \cdot \nabla) \bar{b}$$

$$\bar{b} \times \text{curl } \bar{a} + \bar{a} \times \text{curl } \bar{b} + (\bar{b} \cdot \nabla) \bar{a} + (\bar{a} \cdot \nabla) \bar{b} = \sum i \left[\bar{b} \cdot \frac{\partial \bar{a}}{\partial x} + \bar{a} \cdot \frac{\partial \bar{b}}{\partial x} \right]$$

$$= \sum i \frac{\partial}{\partial x} (\bar{a} \cdot \bar{b})$$

$$= \text{grad} (\bar{a} \cdot \bar{b})$$

$$4) \text{div} (\bar{a} \times \bar{b}) = \bar{b} \cdot \text{curl } \bar{a} - \bar{a} \cdot \text{curl } \bar{b}$$

$$\nabla \cdot (\bar{a} \times \bar{b}) = \bar{b} \cdot (\nabla \times \bar{a}) - \bar{a} \cdot (\nabla \times \bar{b})$$

$$\text{div} (\bar{a} \times \bar{b}) = \sum \bar{i} \cdot \frac{\partial}{\partial x} (\bar{a} \times \bar{b})$$

$$= \sum \bar{i} \cdot \left[\frac{\partial \bar{a}}{\partial x} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial x} \right]$$

$$= \sum \bar{i} \cdot \left(\frac{\partial \bar{a}}{\partial x} \times \bar{b} \right) + \sum \bar{i} \cdot \left(\bar{a} \times \frac{\partial \bar{b}}{\partial x} \right)$$

$$= \sum \left(\bar{i} \times \frac{\partial \bar{a}}{\partial x} \right) \cdot \bar{b} + \sum \bar{i} \cdot \left(\frac{\partial \bar{b}}{\partial x} \times \bar{a} \right)$$

$$= \left(\sum \bar{i} \times \frac{\partial \bar{a}}{\partial x} \right) \cdot \bar{b} - \left(\sum \bar{i} \times \frac{\partial \bar{b}}{\partial x} \right) \cdot \bar{a}$$

$$= \bar{b} \cdot \text{curl } \bar{a} - \text{curl } \bar{b} \cdot \bar{a}$$

$$= \bar{b} \cdot \text{curl } \bar{a} - \bar{a} \cdot \text{curl } \bar{b}$$

$$5) \text{curl} (\bar{a} \times \bar{b}) = \bar{a} \text{div } \bar{b} - \bar{b} \text{div } \bar{a} + (\bar{b} \cdot \nabla) \bar{a} - (\bar{a} \cdot \nabla) \bar{b}$$

$$\text{curl} (\bar{a} \times \bar{b}) = \sum \bar{i} \times \frac{\partial}{\partial x} (\bar{a} \times \bar{b})$$

$$= \sum \bar{i} \times \left[\frac{\partial \bar{a}}{\partial x} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial x} \right]$$

$$= \sum \left[\bar{i} \times \left(\frac{\partial \bar{a}}{\partial x} \times \bar{b} \right) + \sum \bar{i} \times \left(\bar{a} \times \frac{\partial \bar{b}}{\partial x} \right) \right]$$

$$= \sum \left[(\bar{i} \cdot \bar{b}) \frac{\partial \bar{a}}{\partial x} - (\bar{i} \cdot \frac{\partial \bar{a}}{\partial x}) \bar{b} \right] +$$

$$\sum \left[(\bar{i} \cdot \frac{\partial \bar{b}}{\partial x}) \bar{a} - (\bar{i} \cdot \bar{a}) \frac{\partial \bar{b}}{\partial x} \right]$$

$$= \sum (\bar{i} \cdot \bar{b}) \frac{\partial \bar{a}}{\partial x} - \sum (\bar{i} \cdot \frac{\partial \bar{a}}{\partial x}) \bar{b} + \sum (\bar{i} \cdot \frac{\partial \bar{b}}{\partial x}) \bar{a} -$$

$$\sum (\bar{i} \cdot \bar{a}) \frac{\partial \bar{b}}{\partial x}$$

$$= \left(\sum (\bar{i} \frac{\partial}{\partial x}) \cdot \bar{b} \right) \bar{a} - (\text{div } \bar{a}) \bar{b} + (\text{div } \bar{b}) \bar{a} - \left(\sum (\bar{i} \frac{\partial}{\partial x}) \cdot \bar{a} \right) \bar{b}$$

$$= (\nabla \cdot \bar{b}) \bar{a} - \bar{b} \text{div } \bar{a} + \bar{a} \text{div } \bar{b} - \bar{a} (\nabla \cdot \bar{a})$$

$$= \bar{a} \text{div } \bar{b} - \bar{b} \text{div } \bar{a} + (\bar{b} \cdot \nabla) \bar{a} - (\bar{a} \cdot \nabla) \bar{b}$$

$$6) \text{curl}(\text{grad } \phi) = 0$$

$$\boxed{\nabla \times \nabla \phi = 0}$$

$$= \hat{i} \times \frac{\partial}{\partial x} (\nabla \phi) = \hat{i} \frac{\partial}{\partial x} \left(\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \right)$$

$$= \hat{i} \times \frac{\partial}{\partial x} \left[\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \right]$$

$$= \hat{i} \times \left[\hat{i} \frac{\partial^2 \phi}{\partial x^2} + \hat{j} \frac{\partial^2 \phi}{\partial x \partial y} + \hat{k} \frac{\partial^2 \phi}{\partial x \partial z} \right]$$

$$\text{curl}(\text{grad } \phi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \hat{i} \left[\frac{\partial^2 \phi}{\partial y \partial x} - \frac{\partial^2 \phi}{\partial z \partial y} \right] - \hat{j} \left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} \right]$$

$$+ \hat{k} \left[\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right]$$

$$= \hat{i}[0] + \hat{j}[0] + \hat{k}[0]$$

$$= 0$$

$$7) \text{div}(\text{curl } \vec{f}) = 0$$

$$\text{let } \vec{f} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$$

$$\vec{F} = \text{curl } \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \hat{i} \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right] - \hat{j} \left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right] + \hat{k} \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right]$$

$$= F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

$$\therefore \text{div}(\vec{F}) = \text{div}(\text{curl } \vec{f})$$

$$= \frac{\partial}{\partial x} (F_1) + \frac{\partial}{\partial y} (F_2) + \frac{\partial}{\partial z} (F_3)$$

$$= \frac{\partial}{\partial x} \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right] - \frac{\partial}{\partial y} \left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right] +$$

$$\frac{\partial}{\partial z} \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right]$$

$$= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y}$$

$$= 0$$

$$\therefore \text{div}(\text{curl } \vec{f}) = 0$$

8) If f and g are two scalar point functions then

$$\operatorname{div}(f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g$$

$$\nabla f = \bar{i} \frac{\partial f}{\partial x} + \bar{j} \frac{\partial f}{\partial y} + \bar{k} \frac{\partial f}{\partial z}$$

$$\nabla g = \bar{i} \frac{\partial g}{\partial x} + \bar{j} \frac{\partial g}{\partial y} + \bar{k} \frac{\partial g}{\partial z}$$

$$f \nabla g = f \left[\bar{i} \frac{\partial g}{\partial x} + \bar{j} \frac{\partial g}{\partial y} + \bar{k} \frac{\partial g}{\partial z} \right]$$

$$= \bar{i} f \frac{\partial g}{\partial x} + \bar{j} f \frac{\partial g}{\partial y} + \bar{k} f \frac{\partial g}{\partial z}$$

$$= f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$$

$$\operatorname{div}(f \nabla g) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x} \left(f \frac{\partial g}{\partial x} \right) + \frac{\partial}{\partial y} \left(f \frac{\partial g}{\partial y} \right) + \frac{\partial}{\partial z} \left(f \frac{\partial g}{\partial z} \right)$$

$$= \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} + f \frac{\partial^2 g}{\partial y^2} +$$

$$\frac{\partial f}{\partial z} \frac{\partial g}{\partial z} + f \frac{\partial^2 g}{\partial z^2}$$

$$= f \left[\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} \right] + \frac{\partial f}{\partial x} \frac{\partial g}{\partial x} +$$

$$\frac{\partial f}{\partial y} \frac{\partial g}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial g}{\partial z}$$

$$= f \nabla^2 g + \left[\bar{i} \frac{\partial f}{\partial x} + \bar{j} \frac{\partial f}{\partial y} + \bar{k} \frac{\partial f}{\partial z} \right] \cdot \left[\bar{i} \frac{\partial g}{\partial x} + \bar{j} \frac{\partial g}{\partial y} + \bar{k} \frac{\partial g}{\partial z} \right]$$

$$= f \nabla^2 g + \nabla f \cdot \nabla g$$

10-a-22
9) $\operatorname{curl}(\operatorname{curl} \bar{a}) = \operatorname{grad}(\operatorname{div} \bar{a}) - \nabla^2 \bar{a}$

Consider

$$\operatorname{curl}(\operatorname{curl} \bar{a}) = \sum \bar{i} \times \frac{\partial}{\partial x} (\operatorname{curl} \bar{a})$$

$$= \sum \left[\bar{i} \times \frac{\partial}{\partial x} \left[\bar{i} \times \frac{\partial^2 a}{\partial x^2} + \bar{j} \times \frac{\partial^2 a}{\partial x \partial y} + \bar{k} \times \frac{\partial^2 a}{\partial x \partial z} \right] \right]$$

$$= \sum \left[\bar{i} \times \left(\bar{i} \times \frac{\partial^2 a}{\partial x^2} \right) + \bar{i} \times \left(\bar{j} \times \frac{\partial^2 a}{\partial x \partial y} \right) + \bar{i} \times \left(\bar{k} \times \frac{\partial^2 a}{\partial x \partial z} \right) \right]$$

$$= \sum \left[\left(\bar{i} \cdot \frac{\partial^2 a}{\partial x^2} \right) \bar{i} - \left(\bar{i} \cdot \bar{i} \right) \frac{\partial^2 a}{\partial x^2} + \left(\bar{i} \cdot \frac{\partial^2 a}{\partial x \partial y} \right) \bar{j} + \right.$$

$$\left. \left(\bar{i} \cdot \frac{\partial^2 a}{\partial x \partial z} \right) \bar{k} \right] \quad [\bar{i} \cdot (\bar{i} \cdot \bar{j}) = 0, \bar{i} \cdot \bar{k} = 0]$$

$$= \sum \left[\left(\bar{i} \cdot \frac{\partial^2 a}{\partial x^2} \right) \bar{i} + \left(\bar{i} \cdot \frac{\partial^2 a}{\partial x \partial y} \right) \bar{j} + \left(\bar{i} \cdot \frac{\partial^2 a}{\partial x \partial z} \right) \bar{k} \right] - \sum \frac{\partial^2 a}{\partial x^2}$$

$$= \sum \left[\bar{i} \cdot \frac{\partial}{\partial x} \left(\bar{i} \cdot \frac{\partial \bar{a}}{\partial x} \right) + \bar{j} \cdot \frac{\partial}{\partial y} \left(\bar{i} \cdot \frac{\partial \bar{a}}{\partial x} \right) + \bar{k} \cdot \frac{\partial}{\partial z} \left(\bar{i} \cdot \frac{\partial \bar{a}}{\partial x} \right) \right] - \sum \frac{\partial^2 a}{\partial x^2}$$

$$= \sum \operatorname{grad} \left(\bar{i} \cdot \frac{\partial \bar{a}}{\partial x} \right) - \left[\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2} \right]$$

$$= \text{grad} \left(\vec{i} \cdot \frac{\partial \vec{a}}{\partial x} \right) + \nabla^2 \vec{a} \\ = \text{grad} (\text{div} \vec{a}) + \nabla^2 \vec{a}$$

1) Prove that $\nabla f \times \nabla g$ is solenoidal.

To prove $\nabla f \times \nabla g$ is solenoidal it is enough to prove that $\text{div}(\nabla f \times \nabla g) = 0$.

From the known identity (4th identity) we

have

$$\boxed{\text{div}(\vec{a} \times \vec{b}) = \vec{b} \cdot \text{curl} \vec{a} - \vec{a} \cdot \text{curl} \vec{b}}$$

$$\begin{aligned} \therefore \text{div}(\nabla f \times \nabla g) &= \nabla g \cdot \text{curl}(\nabla f) - \nabla f \cdot \text{curl}(\nabla g) \\ &= \nabla g \cdot \vec{0} - \nabla f \cdot \vec{0} \quad [\because \text{curl}(\nabla \phi = 0)] \\ &= 0. \quad (\text{6th identity}) \end{aligned}$$

Note: **

If $\vec{a}$ and $\vec{b}$ are irrotational then $\vec{a} \times \vec{b}$ is always a solenoidal vector.

2) Prove that $\text{div}[(\vec{r} \times \vec{a}) \times \vec{b}] = -2(\vec{b} \cdot \vec{a})$ where $\vec{a}$ and $\vec{b}$ are constant vectors.

$$\therefore (\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$$

$$\text{div}[(\vec{r} \times \vec{a}) \times \vec{b}] = \text{div}[(\vec{b} \cdot \vec{r})\vec{a} - (\vec{b} \cdot \vec{a})\vec{r}]$$

$$\boxed{\text{div}(\vec{r} \vec{a}) = \text{grad} \phi \cdot \vec{a} + \phi \text{div} \vec{a}}$$

$$= \text{grad}(\vec{b} \cdot \vec{r}) \cdot \vec{a} + (\vec{b} \cdot \vec{r}) \text{div} \vec{a} - [\text{grad}(\vec{b} \cdot \vec{a}) \cdot \vec{r} + (\vec{b} \cdot \vec{a}) \text{div} \vec{r}]$$

$$= \text{grad}(\vec{b} \cdot \vec{r}) \cdot \vec{a} + (\vec{b} \cdot \vec{r})(0) - 0 - (\vec{b} \cdot \vec{a})(3)$$

$$[\because \text{div} \vec{a} = 0, \text{div} \vec{r} = 3, \text{grad}(\vec{a} \cdot \vec{b}) = 0 \text{ (}\vec{a}, \vec{b} \text{ constant)}]$$

$$= \text{grad}(\vec{b} \cdot \vec{r}) \cdot \vec{a}$$

$$= \sum_i \vec{i} \frac{\partial}{\partial x_i} (\vec{b} \cdot \vec{r}) \cdot \vec{a} = 3(\vec{b} \cdot \vec{a})$$

$$= \sum_i \vec{i} \left[\frac{\partial \vec{b}}{\partial x_i} \cdot \vec{r} + \vec{b} \cdot \frac{\partial \vec{r}}{\partial x_i} \right] \cdot \vec{a} = 3(\vec{b} \cdot \vec{a})$$

$$= \sum_i \vec{i} (\vec{b} \cdot \vec{i}) \cdot \vec{a} = 3(\vec{b} \cdot \vec{a})$$

$$= 3(\vec{b} \cdot \vec{a}) \cdot \vec{a} = 3(\vec{b} \cdot \vec{a})$$

$$= (\vec{b} \cdot \vec{a}) - 3(\vec{b} \cdot \vec{a}) = -2(\vec{b} \cdot \vec{a})$$

$$1) \rho \cdot \nabla \text{curl}(\vec{r} \times \vec{a}) \times \vec{b} = \vec{b} \times \vec{a}$$

$\nabla \vec{a}$ is a constant vector, $\rho \cdot \nabla$

$$2) \nabla(\vec{a} \cdot \vec{f}) = (\vec{a} \cdot \nabla) \vec{f} + (\vec{a} \times \text{curl} \vec{f})$$

$$3) \nabla \cdot (\vec{a} \times \vec{f}) = -\vec{a} \cdot \text{curl} \vec{f}$$

$$4) \nabla \times (\vec{a} \times \vec{f}) = \vec{a} \text{div} \vec{f} - (\vec{a} \cdot \nabla) \vec{f}$$

$$5) \nabla f = x^2 + y^2 + z^2 \text{ and } \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \text{ s.t.}$$

$$\text{div}(\phi \vec{r}) = 5\phi$$

$$e) \nabla f = x^2 y \vec{i} \text{ and } \vec{g} = xy - z^2 \vec{j} \text{ find}$$

$$(i) \nabla(\nabla f \cdot \nabla g) \quad (ii) \nabla \cdot (\nabla f \times \nabla g)$$

$$1) \rho \cdot \nabla \text{curl}(\vec{r} \times \vec{a}) \times \vec{b} = \vec{b} \times \vec{a}$$

$$\therefore (\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a}$$

$$\text{curl}[(\vec{r} \times \vec{a}) \times \vec{b}] = \text{curl}[(\vec{b} \cdot \vec{r}) \vec{a} - (\vec{b} \cdot \vec{a}) \vec{r}]$$

$$= \text{curl}[(\vec{b} \cdot \vec{r}) \vec{a}] - \text{curl}[(\vec{b} \cdot \vec{a}) \vec{r}]$$

$$\boxed{\text{curl}(\phi \vec{a}) = \text{grad} \phi \times \vec{a} + \phi \text{curl} \vec{a}}$$

$$= \text{grad}(\vec{b} \cdot \vec{r}) \times \vec{a} + (\vec{b} \cdot \vec{r}) \text{curl} \vec{a} - [\text{grad}(\vec{b} \cdot \vec{a}) \times \vec{r} + (\vec{b} \cdot \vec{a}) \text{curl} \vec{r}]$$

$$= \text{grad}(\vec{b} \cdot \vec{r}) \times \vec{a} + (\vec{b} \cdot \vec{r}) \text{curl} \vec{a} \quad [\because \text{grad}(\vec{a} \cdot \vec{b}) = 0 \text{ curl} \vec{r} = 0]$$

$$= \text{grad}(\vec{b} \cdot \vec{r}) \times \vec{a} \quad [\because \text{curl} \vec{a} = 0]$$

$$= \sum i \frac{\partial}{\partial x} (\vec{b} \cdot \vec{r}) \times \vec{a}$$

$$= \sum i \left[\frac{\partial \vec{b}}{\partial x} \cdot \vec{r} + \vec{b} \cdot \frac{\partial \vec{r}}{\partial x} \right] \times \vec{a}$$

$$= \sum i (\vec{b} \cdot \vec{i}) \times \vec{a}$$

$$= \sum i (b_i \vec{i}) \times \vec{a} = \vec{b} \times \vec{a}$$

$$2) \nabla(\vec{a} \cdot \vec{f}) = (\vec{a} \cdot \nabla) \vec{f} + (\vec{a} \times \text{curl} \vec{f})$$

$$\boxed{\text{grad}(\vec{a} \cdot \vec{b}) = (\vec{b} \cdot \nabla) \vec{a} + (\vec{a} \cdot \nabla) \vec{b} + \vec{b} \times \text{curl} \vec{a} + \vec{a} \times \text{curl} \vec{b}}$$

$$\nabla(\vec{a} \cdot \vec{f}) = (\vec{f} \cdot \nabla) \vec{a} + (\vec{a} \cdot \nabla) \vec{f} + \vec{f} \times \text{curl} \vec{a} + \vec{a} \times \text{curl} \vec{f}$$

$$= (\vec{a} \cdot \nabla) \vec{f} + (\vec{a} \times \text{curl} \vec{f}) \quad [\because \text{curl} \vec{a} = 0,$$

$(\vec{f} \cdot \nabla) \vec{a} = 0$ ($\vec{a}$ is constant vector)]

$$3) \nabla \cdot (\bar{a} \times \bar{f}) = -\bar{a} \text{curl} \bar{f}$$

$$\Rightarrow \text{div}(\bar{a} \times \bar{f})$$

$$\boxed{\text{div}(\bar{a} \times \bar{b}) = \bar{b} \cdot \text{curl} \bar{a} - \bar{a} \cdot \text{curl} \bar{b}}$$

$$\text{div}(\bar{a} \times \bar{f}) = \bar{f} \cdot \text{curl} \bar{a} - \bar{a} \text{curl} \bar{f}$$

$$\text{div}(\bar{a} \times \bar{f}) = -\bar{a} \cdot \text{curl} \bar{f} \quad (\because \text{curl} \bar{a} = 0)$$

$$4) \nabla \times (\bar{a} \times \bar{f}) = \bar{a} \text{div} \bar{f} - (\bar{a} \cdot \nabla) \bar{f}$$

$$\text{curl}(\bar{a} \times \bar{f}) = \bar{a} \text{div} \bar{f} - \bar{f} \text{div} \bar{a} + (\bar{f} \cdot \nabla) \bar{a} - (\bar{a} \cdot \nabla) \bar{f}$$

$$= \bar{a} \text{div} \bar{f} + (\bar{f} \cdot \nabla) \bar{a} - (\bar{a} \cdot \nabla) \bar{f} \quad [\because \text{div} \bar{a} = 0]$$

$$= \bar{a} \text{div} \bar{f} - (\bar{a} \cdot \nabla) \bar{f} \quad [\because (\bar{f} \cdot \nabla) \bar{a} = 0]$$

$$5) \phi = x^2 y^2 + z^2 \quad \bar{e}, \bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \quad \text{P.T. } \text{div}(\phi \bar{r}) = 5\phi$$

$$\text{div}(\phi \bar{r}) = \text{grad} \phi \cdot \bar{r} + \phi \text{div} \bar{r}$$

$$= \text{grad}(x^2 y^2 + z^2) \cdot \bar{r} + (x^2 + y^2 + z^2) 3 \quad [\because \text{div} \bar{r} = 3]$$

$$= 2x \frac{\partial}{\partial x} (x^2 y^2 + z^2) \cdot \bar{r} + 3(x^2 + y^2 + z^2)$$

$$= 2x [2x] \cdot \bar{r} + 3(x^2 + y^2 + z^2)$$

$$= 2x^2 \bar{r} \cdot \bar{r} + 3(x^2 + y^2 + z^2)$$

$$= 2\bar{r} \cdot \bar{r} + 3(x^2 + y^2 + z^2) \quad [\because \bar{r} \cdot \bar{r} = x^2 + y^2 + z^2]$$

$$= 2(x^2 + y^2 + z^2) + 3(x^2 + y^2 + z^2)$$

$$= 5(x^2 + y^2 + z^2)$$

$$= 5\phi$$

$$6) f = x^2 yz \quad \& \quad g = xy - 3z^2$$

$$i) \nabla(f \cdot \nabla g)$$

$$\nabla f = \text{grad} f = i \frac{\partial}{\partial x} (x^2 yz) + j \frac{\partial}{\partial y} (x^2 yz) +$$

$$k \frac{\partial}{\partial z} (x^2 yz)$$

$$= i 2xy z + j x^2 z + k x^2 y$$

$$\nabla g = \text{grad} g = i \frac{\partial}{\partial x} (xy - 3z^2) + j \frac{\partial}{\partial y} (xy - 3z^2) +$$

$$k \frac{\partial}{\partial z} (xy - 3z^2)$$

$$= i(y) + j(x) + k(0 - 6z)$$

$$\nabla f \cdot \nabla g = 2xy^2 z + x^2 z + (-6x^2 yz)$$

$$\nabla(f \cdot \nabla g) = i \frac{\partial}{\partial x} [2xy^2 z] +$$

$$j \frac{\partial}{\partial y} (2xy^2 z + x^2 z - 6x^2 yz) +$$

$$k \frac{\partial}{\partial z} (2xy^2 z + x^2 z - 6x^2 yz)$$

$$= i(2y^2z + 3x^2z - 12xyz) + j(4xy^2 - 6x^2z) + k(2xy^2 + x^3 - 6x^2y)$$

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Directional Derivative:-

Normal vector: If ϕ is any scalar point function (surface) then the normal vector along ϕ is given by $\nabla\phi$ or $\text{grad}\phi$ and its unit normal vector along ϕ is $\frac{\nabla\phi}{|\nabla\phi|}$ i.e.

$$\bar{e} = \frac{\nabla\phi}{|\nabla\phi|}$$

Directional Derivative:- The directional derivative of a scalar point function ϕ at point p in the direction of a vector point function $\bar{f}$ is given by $\bar{e} \cdot \nabla\phi$ where ' $\bar{e}$ ' is unit normal vector along $\bar{f}$. i.e.

$$\bar{e} = \frac{\bar{f}}{|\bar{f}|}$$

Angle between two Surfaces:-

If θ is the angle between two surfaces

$$\phi_1 \text{ and } \phi_2 \text{ then } \cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| |\nabla\phi_2|}$$

Note:-

If ϕ is any scalar point function then along the direction of $\nabla\phi$ the directional derivative is maximum, and the maximum value of

directional derivative of point P is $|\nabla \phi|_P$

1) Find the unit normal vector at the point $(1, -1, 2)$ to the surface $x^2y + y^2z + z^2x = 5$.

Given surface

$$x^2y + y^2z + z^2x = 5$$

Now the normal vector $\vec{n}$ to the given surface ϕ is $\nabla \phi$ i.e. $\vec{n} = \nabla \phi$.

At $(1, -1, 2)$

$$\vec{n} = \vec{i}(-2+4) + \vec{j}(1-4) + \vec{k}(1+4)$$

$$\vec{n} = 2\vec{i} - 3\vec{j} + 5\vec{k}$$

$$\therefore \vec{e} = \text{unit normal vector} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\vec{i} - 3\vec{j} + 5\vec{k}}{\sqrt{4+9+25}}$$

$$= \frac{2\vec{i} - 3\vec{j} + 5\vec{k}}{\sqrt{38}}$$

2) Find the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction of vector $2\vec{i} - \vec{j} - 2\vec{k}$

$$\text{G.T. } \phi(x, y, z) = x^2yz + 4xz^2, P = (1, -2, -1)$$

$$\vec{f} = 2\vec{i} - \vec{j} - 2\vec{k}$$

$$\text{Now } \nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$= \vec{i}(2xyz + 4z^2) + \vec{j}(x^2z) + \vec{k}(x^2y + 8xz)$$

At point $P = (1, -2, -1)$

$$|\nabla \phi|_P = \vec{i}(4+4) + \vec{j}(-1) + \vec{k}(-2-8)$$

$$= 8\vec{i} - \vec{j} - 10\vec{k}$$

$$\vec{e} = \frac{\vec{f}}{|\vec{f}|} = \frac{2\vec{i} - \vec{j} - 2\vec{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\vec{i} - \vec{j} - 2\vec{k})$$

$\therefore$ The directional derivative of $\phi(x, y, z)$ at point $P(1, -2, -1)$ in the direction of vector $\vec{f}$ is $\vec{e} \cdot \nabla \phi$.

$$= \frac{1}{3}(2\vec{i} - \vec{j} - 2\vec{k}) \cdot (8\vec{i} - \vec{j} - 10\vec{k}) + \vec{k}(x^2y + 8xz)$$

$$= \frac{2}{3} + 2 = \frac{1}{3}(2\vec{i} - \vec{j} - 2\vec{k}) \cdot (8\vec{i} - \vec{j} - 10\vec{k})$$

$$= \left(\frac{2}{3} \cdot 8\right) + \left(-\frac{1}{3} \cdot -1\right) + \left(-\frac{2}{3} \cdot -10\right)$$

$$= \frac{16}{3} + \frac{1}{3} + \frac{20}{3}$$

$$= \frac{1}{3}[16+1+20] = \frac{37}{3}$$

3) Find the directional derivative of $\phi(x, y, z)$ at the point $(-1, 2, 3)$ in the direction towards the point $(2, -1, -1)$.

G.T surface $\phi(x, y, z) = x^4 + y^4 + z^4$ and scalar point

$P = (-1, 2, 3)$ and $\bar{f}$ is the normal vector along PA where $A = (2, -1, -1)$.

$$\overline{OP} = -\bar{i} + 2\bar{j} + 3\bar{k}$$

$$\overline{OA} = 2\bar{i} - \bar{j} - \bar{k}$$

$$\bar{f} = \overline{PA} = \overline{OA} - \overline{OP}$$

$$\bar{f} = 3\bar{i} - 3\bar{j} - 4\bar{k}$$

$$\bar{e} = \frac{\bar{f}}{|\bar{f}|} = \frac{3\bar{i} - 3\bar{j} - 4\bar{k}}{\sqrt{9+9+16}} = \frac{3\bar{i} - 3\bar{j} - 4\bar{k}}{\sqrt{34}}$$

$$\begin{aligned} \nabla \phi &= \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z} \\ &= \bar{i}(4x^3) + \bar{j}(4y^3) + \bar{k}(4z^3) \end{aligned}$$

At point $(-1, 2, 3)$

$$|\nabla \phi|_P = -4\bar{i} + 32\bar{j} + 108\bar{k}$$

$\therefore$ Directional derivative of a scalar point ϕ at point P in direction of vector PA is $\bar{e} \cdot \nabla \phi$

$$\Rightarrow \frac{3\bar{i} - 3\bar{j} - 4\bar{k}}{\sqrt{34}} \cdot (-4\bar{i} + 32\bar{j} + 108\bar{k})$$

$$\Rightarrow \frac{1}{\sqrt{34}} [-12 - 96 - 432] = \frac{-540}{\sqrt{34}} \quad (\text{ignore -ve sign})$$

4) Find the directional derivative of $xyz^2 + xz$ at $(1, 1, 1)$ in direction of the normal to the surface $3xy^2 + y + z$ at $(0, 1, 1)$

G.T $\phi_1 = xyz^2 + xz$ and $P = (1, 1, 1)$.

Now we have to find the normal vector to the surface $\phi_2 = 3xy^2 + y + z$ at the point

$Q(0, 1, 1)$

$$\phi_2 = 3xy^2 + y + z, \quad Q = (0, 1, 1)$$

$$\bar{n} = \bar{f} = |\nabla \phi_2|_Q$$

$$= \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$$

$$= \bar{i}(3y^2) + \bar{j}(6xy + 1) + \bar{k}(1)$$

at $Q = (0, 1, 1)$

$$[\nabla \phi_2]_a = \bar{i}(s) + \bar{j}(1) + \bar{k}(1)$$

$$= 3\bar{i} + \bar{j} + \bar{k}$$

Now, unit normal vector $\bar{e} = \frac{\bar{j}}{|\bar{j}|}$

$$\bar{e} = \frac{3\bar{i} + \bar{j} + \bar{k}}{\sqrt{9+1+1}} = \frac{3\bar{i} + \bar{j} + \bar{k}}{\sqrt{11}}$$

Now $\nabla \phi_1$ at P is

$$\nabla \phi_1 = \bar{i} \frac{\partial \phi_1}{\partial x} + \bar{j} \frac{\partial \phi_1}{\partial y} + \bar{k} \frac{\partial \phi_1}{\partial z}$$

$$= \bar{i}(y z^2 + z) + \bar{j}(x z^2) + \bar{k}(2xy z + 2k)$$

$\nabla \phi_1|_P$ at $P(1,1,1)$

$$= \bar{i}(1+1) + \bar{j}(1) + \bar{k}(2+1)$$

$$= 2\bar{i} + \bar{j} + 3\bar{k}$$

$\therefore$ Directional derivative along the ϕ_1 and point P along normal vector is

$$= \bar{e} \cdot \nabla \phi_1|_P$$

$$= \frac{1}{\sqrt{11}} (3\bar{i} + \bar{j} + \bar{k}) \cdot (2\bar{i} + \bar{j} + 3\bar{k})$$

$$= \frac{1}{\sqrt{11}} [6+1+3] = \frac{10}{\sqrt{11}}$$

5) In what direction from the point $(2,1,-1)$ is the directional derivative of $\phi(x,y,z) = x^2 y z^3$ a maximum? What is its magnitude?

Sol:- Since the directional derivative is max

along $\nabla \phi|_P$

$$\text{Now } \nabla \phi|_P = x^2 y z^2$$

$$= \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$$

$$= \bar{i}(2xy z^3) + \bar{j}(x^2 z^3) + \bar{k}(3x^2 y z^2)$$

At $P = (2,1,-1)$

$$\nabla \phi|_P = \bar{i}(-4) + \bar{j}(-2(1)) + \bar{k}(3(4)(1)(-1))$$

$$= -4\bar{i} - 2\bar{j} + 12\bar{k}$$

$$|\nabla \phi|_P = \sqrt{16+16+144}$$

$$= \sqrt{176}$$

6) Find the angle between the normal to the surface $xy = z^2$ at $(1,4,2)$ and $(-3, -3, 3)$.

Sol:- Given surface is $xy - z^2 = \phi$

$$\text{Now } \nabla \phi = \bar{i} \frac{\partial \phi}{\partial x} + \bar{j} \frac{\partial \phi}{\partial y} + \bar{k} \frac{\partial \phi}{\partial z}$$

$$= y\bar{i} + \bar{j}(x) + \bar{k}(-2z) = y\bar{i} + x\bar{j} - 2z\bar{k}$$

Now consider $\vec{n}_1$ and $\vec{n}_2$ are the normal vectors to the given surface at $P = (1, 4, 2)$ and $Q = (-3, -3, 3)$

$$\vec{n}_1 = \nabla \phi|_P = \vec{i}(4) + \vec{j}(1) + \vec{k}(-2(2))$$

$$= 4\vec{i} + \vec{j} - 4\vec{k}$$

$$\vec{n}_2 = \nabla \phi|_Q = \vec{i}(-3) + \vec{j}(-3) + \vec{k}(-2(3))$$

$$= -3\vec{i} - 3\vec{j} - 6\vec{k}$$

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{(-12 - 3 + 24)}{\sqrt{16+1+16} \sqrt{9+9+36}}$$

$$= \frac{9}{\sqrt{33}\sqrt{54}} = \frac{9}{\sqrt{1782}}$$

$$\therefore \theta = \cos^{-1} \left(\frac{9}{\sqrt{1782}} \right) = 1.35$$

Probs:-

$$1) \operatorname{div}(\phi \vec{a}) = \operatorname{grad} \phi \cdot \vec{a} + \phi \operatorname{div} \vec{a}$$

$$\cdot \operatorname{div}(\phi \vec{a}) = \sum \vec{i} \cdot \frac{\partial}{\partial x} (\phi \vec{a})$$

$$= \sum \vec{i} \cdot \frac{\partial}{\partial x} \left[\phi \frac{\partial \vec{a}}{\partial x} + \vec{a} \frac{\partial \phi}{\partial x} \right]$$

$$= \sum \vec{i} \cdot \left[\phi \frac{\partial \vec{a}}{\partial x} + \vec{a} \frac{\partial \phi}{\partial x} \right]$$

$$= \sum \vec{i} \cdot \left(\phi \frac{\partial \vec{a}}{\partial x} \right) + \sum \vec{i} \cdot \left(\vec{a} \frac{\partial \phi}{\partial x} \right)$$

$$= \left(\sum \vec{i} \cdot \frac{\partial \phi}{\partial x} \right) \cdot \vec{a} + \left(\sum \vec{i} \cdot \vec{a} \frac{\partial \phi}{\partial x} \right)$$

$$= \operatorname{grad} \phi \cdot \vec{a} + \phi \operatorname{div} \vec{a}$$

7) Find the directional derivative of $\phi = xy + yz + zx$

at A in the direction of $\vec{AB}$ where $\vec{A} = (1, 2, -1)$

$$\vec{B} = (1, 2, 3)$$

Given that the surface $\phi = xy + yz + zx$

$$\text{at } A = (1, 2, -1)$$

$$\vec{f} = ? \quad \vec{B} = (1, 2, 3)$$

$$\vec{OA} = \vec{i} + 2\vec{j} - \vec{k} \quad \vec{OB} = \vec{i} + 2\vec{j} + 3\vec{k}$$

$$\vec{f} = \vec{AB} = \vec{OB} - \vec{OA} = (1-1)\vec{i} + (2-2)\vec{j} + (3+1)\vec{k} = 4\vec{k}$$

$$e = \frac{\vec{f}}{|\vec{f}|} = \frac{4\vec{k}}{\sqrt{4}} = \frac{4\vec{k}}{2}$$

The normal $\vec{n} = \nabla \phi$

$$\phi = xy + yz + xz$$

$$\vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} = (y+z)\vec{i} + (x+z)\vec{j} + (y+x)\vec{k}$$

$$\text{At } A(1, 2, -1) = \vec{i} + 2\vec{j} + 3\vec{k}$$

The directional derivative $= \vec{e} \cdot \nabla \phi|_P$

$$= \frac{(4\vec{k}) \cdot (\vec{i} + 0\vec{j} + 3\vec{k})}{4}$$

$$= \frac{12}{4} = 3$$

8) Find the direction derivative of $\phi = x^2 - 2y^2 + 4xz$ at $(1, 1, -1)$ in the direction of $2\vec{i} + \vec{j} - \vec{k}$

$$\phi = x^2 - 2y^2 + 4xz \text{ at } P = (1, 1, -1)$$

$$\vec{f} = 2\vec{i} + \vec{j} - \vec{k}$$

$$\text{The normal vector } \vec{n} = \nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$= \vec{i}(2x) + \vec{j}(-4y) + \vec{k}(4z)$$

$$\text{At } P = (1, 1, -1) \cdot \nabla \phi|_P = 2\vec{i} - 4\vec{j} + 8\vec{k}$$

$$\vec{e} = \frac{\vec{f}}{|\vec{f}|} = \frac{2\vec{i} + \vec{j} - \vec{k}}{\sqrt{4+1+1}} = \frac{2\vec{i} + \vec{j} - \vec{k}}{\sqrt{6}}$$

$\therefore$ The D.D is $\vec{e} \cdot \nabla \phi|_P$

$$= \frac{1}{\sqrt{6}} (2\vec{i} + \vec{j} - \vec{k}) \cdot (2\vec{i} - 4\vec{j} - 8\vec{k})$$

$$= \frac{4-4+8}{\sqrt{6}} = \frac{8}{\sqrt{6}}$$

9) Find the unit normal vector to the surface

$$x^3 + y^3 + 3xyz = 3 \text{ at pt } P = (1, 2, -1)$$

$$\text{Given } \phi(x, y, z) = x^3 + y^3 + 3xyz - 3$$

$$\text{at } P = (1, 2, -1)$$

$$\text{The normal vector } \vec{n} = \nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$= (3x^2 + 3yz)\vec{i} + (3y^2 + 3xz)\vec{j} + (3xy)\vec{k}$$

$$\text{At } P = (1, 2, -1) = (3-6)\vec{i} + (12-3)\vec{j} + 6\vec{k}$$

$$\vec{n} = -3\vec{i} + 9\vec{j} + 6\vec{k}$$

$$\text{The normal vector} = \frac{\vec{n}}{|\vec{n}|} = \frac{-3\vec{i} + 9\vec{j} + 6\vec{k}}{\sqrt{9+81+36}}$$

$$= \frac{-3\vec{i} + 9\vec{j} + 6\vec{k}}{\sqrt{126}}$$

$$= \frac{+3(-\hat{i} + 3\hat{j} + 2\hat{k})}{\sqrt{3^2 + 3^2 + 2^2}} = \frac{-\hat{i} + 3\hat{j} + 2\hat{k}}{\sqrt{14}}$$

VECTOR INTEGRATION:-

Closed Curve:- Let 'C' be a curve in a space. Let A be the initial point and B be the terminal point in curve 'C'.

Where the direction along C oriented from A to B is positive and the direction from B to A is called negative direction. If the two points A and B coincide then the curve C is called the closed curve.

Line Integral:- Any integral which is evaluated along the curve is called line integral and is denoted by $\int_C \vec{F} \cdot d\vec{r}$ when $\vec{F}$ is a vector point function and $\vec{r}$ is position vector. i.e.

$$\text{if } \vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k} \text{ and } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{then } d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}.$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C (f_1\hat{i} + f_2\hat{j} + f_3\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ = \int_C f_1 dx + f_2 dy + f_3 dz.$$

CIRCULATION:- If C is a simple closed curve then the integral of $\vec{F}$ around C is called the circulation of $\vec{F}$ about C and is denoted by $\oint_C \vec{F} \cdot d\vec{r}$

Note:-

$\Rightarrow$ If $\oint_C \vec{F} \cdot d\vec{r} = 0$ then $\vec{F}$ is called conservative force field / field vector. i.e., no work is done and the energy is conserved.

$\Rightarrow$ If the circulation of $\vec{F}$ around every closed curve in a region vanishes then $\vec{F}$ is said to be irrotational.

20-9-22

WORK DONE BY A FORCE:-

$\Rightarrow$ If $\vec{F}$ represents the force vector acting on a particle moving along an arc AB then the work done during a displacement $\vec{s}$ is $\vec{F} \cdot \vec{s}$.

Hence the total work done by $\vec{F}$ during the displacement from A to B is given by the line integral $\int_A^B \vec{F} \cdot d\vec{r}$.

$$= \frac{+3(-\hat{i} + 3\hat{j} + 2\hat{k})}{\sqrt{10 \times 14}} = \frac{-\hat{i} + 3\hat{j} + 2\hat{k}}{\sqrt{14}} //$$

VECTOR INTEGRATION:-

Closed Curve:- Let 'C' be a curve in a space. Let A be the initial point and B be the terminal point in curve 'C'.

Where the direction along C oriented from A to B is positive and the direction from B to A is called negative direction.

If the two points A and B coincide then the curve C is called the closed curve.

Line Integral:- Any integral which is evaluated along the curve is called line integral and is denoted by $\int_C \vec{F} \cdot d\vec{r}$ when $\vec{F}$ is a vector point function and $\vec{r}$ is position vector. i.e.

if $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
then $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$.

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C (f_1\hat{i} + f_2\hat{j} + f_3\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C f_1 dx + f_2 dy + f_3 dz.$$

CIRCULATION:- If C is a simple closed curve then the integral of $\vec{F}$ around C is called the circulation of $\vec{F}$ about C and is denoted by $\oint_C \vec{F} \cdot d\vec{r}$

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No work is done and the energy is conserved

$\Rightarrow$ If the circulation of $\vec{F}$ around every closed curve in a region vanishes then $\vec{F}$ is said to be irrotational.

20-9-22

WORK DONE BY A FORCE:-

$\Rightarrow$ If $\vec{F}$ represents the force vector acting on a particle moving along an arc AB then the work done during a ^{small} displacement $d\vec{r}$ is $\vec{F} \cdot d\vec{r}$.

Hence the total work done by $\vec{F}$ during displacement from A to B is given by the line integral $\int_A^B \vec{F} \cdot d\vec{r}$.

Note:-

→ If the force $\vec{F}$ is conservative then the work done is independent of the path.

1) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2\vec{i} + y^3\vec{j}$ and the curve C is the parabola $y = x^2$ in the xy plane from $(0,0)$ to $(1,1)$.

Sol) Given,

$$\vec{F} = x^2\vec{i} + y^3\vec{j}$$

$$\text{Let } \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\therefore \vec{F} \cdot d\vec{r} = (x^2\vec{i} + y^3\vec{j}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= x^2 dx + y^3 dy$$

$$\text{G.T } y = x^2$$

$$dy = 2x dx$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{x=0}^1 x^2 dx + (x^6 \cdot 2x) dx$$

$$= \int_0^1 (x^2 + 2x^7) dx$$

$$= \left[\frac{x^3}{3} + \frac{2x^8}{8} \right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

2) If $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the st. line joining $(0,0,0)$ to $(1,1,1)$. Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\vec{F} \cdot d\vec{r} = (3x^2 + 6y) dx - 14yz dy + 20xz^2 dz$$

Equation of line joining two points $(0,0,0)$ & $(1,1,1)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t$$

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t$$

$$x = y = z = t$$

$$\Rightarrow dx = dy = dz = dt$$

$$\text{and } t \rightarrow 0 \text{ to } 1$$

$$\vec{F} \cdot d\vec{r} = (3x^2 + 6y) dx - 14yz dy + 20xz^2 dz$$

$$= [(3t^2 + 6t) - 14t^2 + 20t^3] dt$$

$$= [20t^3 - 11t^2 + 6t] dt$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^1 (20t^3 - 11t^2 + 6t) dt$$

$$= \left[\frac{20t^4}{4} - 11\frac{t^3}{3} + 6\frac{t^2}{2} \right]_0^1$$

$$= 5 - \frac{11}{3} + 3$$

$$= 8 - \frac{11}{3}$$

$$= 13/3 //$$

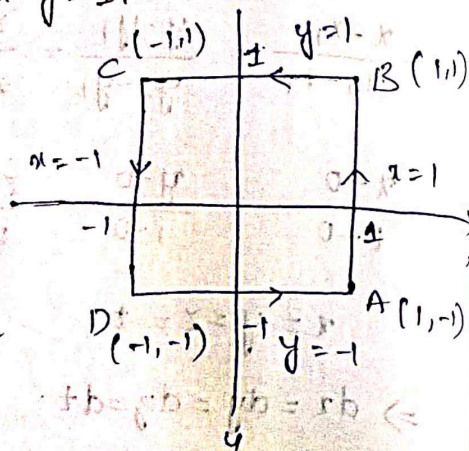
5) Evaluate the line integral $\int_C (x^2 + xy) dx + (x^2 + y^2) dy$ where C is the square formed by the lines $x = \pm 1$ and $y = \pm 1$.

Sol:- The given lines

are $x = \pm 1, y = \pm 1$

Now we have to find the line integral

along AB, BC, CD, DA .



$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CD} \vec{F} \cdot d\vec{r} + \int_{DA} \vec{F} \cdot d\vec{r} \quad \text{--- (1)}$$

Case (i):- [Along AB]

$$x = 1 \Rightarrow dx = 0$$

$$y \rightarrow -1 \text{ to } 1$$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_{y=-1}^1 (1+y)(0) + (1+y^2) dy$$

$$\begin{aligned} &= \int_{-1}^1 (1+y^2) dy = \left[y + \frac{y^3}{3} \right]_{-1}^1 \\ &= \left(1 + \frac{1}{3} \right) - \left(-1 - \frac{1}{3} \right) = 2 + \frac{2}{3} \\ &= 8/3 \end{aligned}$$

Case (ii):- Along BC :-

$$y = 1 \Rightarrow dy = 0$$

$$x \rightarrow 1 \text{ to } -1$$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_{x=1}^{-1} (x^2 + x) dx + (x^2 + 1)(0)$$

$$\begin{aligned} &= \int_{1}^{-1} (x^2 + x) dx = \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_1^{-1} \\ &= \left(\frac{-1}{3} + \frac{1}{2} \right) - \left(\frac{1}{3} + \frac{1}{2} \right) \end{aligned}$$

$$= -\frac{1}{3} + \frac{1}{2} - \frac{1}{3} - \frac{1}{2} = -\frac{2}{3}$$

Case (iii):- Along CD

$$x = -1 \Rightarrow dx = 0$$

$$y \rightarrow 1 \text{ to } -1$$

$$\begin{aligned} \int_{CD} \vec{F} \cdot d\vec{r} &= \int_{y=1}^{-1} (-1-y)(0) + (1+y^2) dy \\ &\Rightarrow \int_{1}^{-1} (1+y^2) dy = \left[y + \frac{y^3}{3} \right]_1^{-1} \end{aligned}$$

$$(-1 + (\frac{1}{3})) - (1 + \frac{1}{3}) = -2 - \frac{2}{3} = -\frac{8}{3}.$$

$$= (-1 + \frac{1}{3}) - (1 + \frac{1}{3}) = -1 + \frac{1}{3} - 1 - \frac{1}{3} = -\frac{8}{3}.$$

Case (iv): - Along DA.

$$y = -1 \Rightarrow dy = 0.$$

$$x \rightarrow -1 \text{ to } 1$$

$$\int_{DA} \vec{F} \cdot d\vec{r} = \int_{x=-1}^1 (x^2 - x) dx + (x^2 - 1)(0)$$

$$= \int_{-1}^1 (x^2 - x) dx = \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_{-1}^1$$

$$= \left[\frac{1}{3} - \frac{1}{2} \right] - \left[-\frac{1}{3} - \frac{1}{2} \right]$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{3} + \frac{1}{2} = \frac{2}{3}.$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CD} \vec{F} \cdot d\vec{r} + \int_{DA} \vec{F} \cdot d\vec{r}$$

$$= \frac{8}{3} - \frac{2}{3} - \frac{8}{3} + \frac{2}{3}$$

$$= 0.$$

4) Find the work done by the force

$$\vec{F} = (3x^2 + 6yz)\vec{i} + (2x + 3xz)\vec{j} + (1 - 4xy - z^2)\vec{k}$$

in moving particle from the point (0,0,0) to (1,1,1) along the curve $C: x=t, y=t^2, z=t^3$.

$$\vec{F} = (3x^2 + 6yz)\vec{i} + (2x + 3xz)\vec{j} + (1 - 4xy - z^2)\vec{k}$$

$$\vec{F} \cdot d\vec{r} = (3x^2 + 6yz)dx + (2x + 3xz)dy + (1 - 4xy - z^2)dz$$

Along curve $x=t, y=t^2, z=t^3$.

$$\vec{F} \cdot d\vec{r} = (3t^2 + 6t^5)dt + (2t + 3t^4)2tdt + (1 - 4t^3 - 3t^6)3t^2dt$$

$$\vec{F} \cdot d\vec{r} = (3t^2 + 6t^5 + 4t^2 + 6t^5 - 12t^5 - 9t^7)dt$$

$$= (-12t^4 + 12t^5 + 10t^2)dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^1 (-12t^4 + 12t^5 + 10t^2)dt$$

$$= \left[-12 \frac{t^5}{5} + 12 \frac{t^6}{6} + \frac{10t^3}{3} \right]_0^1$$

$$= -1 + 2 + \frac{10}{3} = 1 + \frac{10}{3} = \frac{13}{3}.$$

5) Find the circulation of
 $\vec{F} = (2x - y + 2z)\vec{i} + (x + y - z)\vec{j} + (3x - 2y - 5z)\vec{k}$
 along the circle $x^2 + y^2 = 4$ in xy plane.

Sol: $\vec{F} = (2x - y + 2z)\vec{i} + (x + y - z)\vec{j} + (3x - 2y - 5z)\vec{k}$

In xy plane $z = 0$.

$$\vec{F} = (2x - y)\vec{i} + (x + y)\vec{j} + (3x - 2y)\vec{k}$$

$$\vec{F} \cdot d\vec{r} = (2x - y)dx + (x + y)dy + (3x - 2y)dz$$

From circle $x^2 + y^2 = 4 \Rightarrow x^2 + y^2 = r^2$
 $\Rightarrow r = 2$.

$$dx = r \cos \theta, \quad dy = r \sin \theta$$

$$dx = 2 \cos \theta, \quad dy = 2 \sin \theta \Rightarrow dy = 2 \cos \theta$$

$$dz = -2 \sin \theta d\theta$$

$$\vec{F} \cdot d\vec{r} = [2(2 \cos \theta) - 2 \sin \theta](-2 \sin \theta d\theta) +$$

$$[2 \cos \theta + 2 \sin \theta](2 \cos \theta) d\theta$$

$$= [(4 \cos \theta - 2 \sin \theta)(-2 \sin \theta) + (2 \cos \theta + 2 \sin \theta)(2 \cos \theta)] d\theta$$

$$= [-8 \cos \theta \sin \theta + 4 \sin^2 \theta + 4 \cos^2 \theta + 4 \sin \theta \cos \theta] d\theta$$

$$= \int [4 - 4 \sin \theta \cos \theta] d\theta$$

$$= \int [4 - 2(2 \sin \theta \cos \theta)] d\theta$$

$$= \int [4 - 2 \sin 2\theta] d\theta \quad [\because 2 \sin \theta \cos \theta = \sin 2\theta]$$

$$\int \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (4 - 2 \sin 2\theta) d\theta$$

$$= \left[4\theta + \frac{2 \cos 2\theta}{2} \right]_0^{2\pi}$$

$$= 4(2\pi) + \frac{\cos(2\pi)}{1} - \left[0 + \frac{\cos(0)}{1} \right]$$

$$= 8\pi + 1 - 1$$

$$= 8\pi$$

Surface Integrals:-

If S is any surface and $\vec{n}$ is the outward drawn unit normal vector to the surface S then $\int_S \vec{F} \cdot \vec{n} \, ds$ is called the

surface integral.

Note:-

$\rightarrow$ If $\mathcal{R}$ is the projection of S lies on xy plane

$$\text{then } ds = \frac{dx \, dy}{|\vec{n} \cdot \vec{k}|}$$

→ If R is the projection of S lies on yz plane then $ds = \frac{dy dz}{|\vec{n} \cdot \vec{j}|}$

→ If R is the projection of S lies on xz plane then $ds = \frac{dx dz}{|\vec{n} \cdot \vec{j}|}$

Q) Evaluate $\int_S \vec{F} \cdot \vec{n} ds$ where $\vec{F} = 18xz\vec{i} - 12\vec{j} + 3yz\vec{k}$ and S is the part of the surface of the plane $2x + 3y + 6z = 12$ located in the first octant.

Sol:- G.T
 $\phi = 2x + 3y + 6z = 12$

$$\nabla \phi = \vec{i}(2) + \vec{j}(3) + \vec{k}(6)$$

$$= 2\vec{i} + 3\vec{j} + 6\vec{k}$$

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2\vec{i} + 3\vec{j} + 6\vec{k}}{\sqrt{4+9+36}} = \frac{1}{7}(2\vec{i} + 3\vec{j} + 6\vec{k})$$

$$\text{Now } \vec{F} \cdot \vec{n} = (18xz\vec{i} - 12\vec{j} + 3yz\vec{k}) \cdot \frac{1}{7}(2\vec{i} + 3\vec{j} + 6\vec{k})$$

$$= \frac{1}{7}(36z - 36 + 18y)$$

R on xy plane

$$\therefore ds = \frac{dx dy}{|\vec{n} \cdot \vec{k}|}$$

$$\text{Now } |\vec{n} \cdot \vec{k}| = \frac{1}{7}(2\vec{i} + 3\vec{j} + 6\vec{k}) \cdot \vec{k}$$

$$= 6/7$$

$$\therefore ds = \frac{dx dy}{6/7}$$

Limits of x, y :-

In xy plane $z = 0$

$$2x + 3y = 12$$

$$2x = 12 - 3y \Rightarrow x = \frac{12 - 3y}{2}$$

$$\therefore x \rightarrow 0 \text{ to } \frac{12 - 3y}{2}$$

$$\text{If } x = 0, 3y = 12$$

$$\boxed{y = 4}$$

$$\therefore y \rightarrow 0 \text{ to } 4$$

Given surface is $2x + 3y + 6z = 12$

$$6z = 12 - 2x - 3y$$

$$z = \frac{1}{6}(12 - 2x - 3y) \quad \text{--- (2)}$$

Sub (2) in (1)

$$\therefore \vec{F} \cdot \vec{n} = \frac{1}{7} \left[\frac{36}{6}(12 - 2x - 3y) - 36 + 18y \right]$$

$$\begin{aligned}
 &= \frac{1}{7} [72 - 12x - 12y - 36 + 12xy] \\
 &= \frac{1}{7} [36 - 12x] \\
 &= \frac{12}{7} [3 - x]
 \end{aligned}$$

$$\therefore \int_S \vec{F} \cdot \vec{n} \, ds = \int_{y=0}^4 \int_{x=0}^{\frac{12-3y}{2}} \frac{12}{7} (3-x) \, dx \, dy$$

$$= 2 \int_{y=0}^4 \left[3x - \frac{x^2}{2} \right]_0^{\frac{12-3y}{2}} dy$$

$$= 2 \int_{y=0}^4 \left[3 \left(\frac{12-3y}{2} \right) - \frac{1}{2} \left(\frac{12-3y}{2} \right)^2 \right] dy$$

$$= 2 \int_{y=0}^4 \left[\frac{9}{2} (4-y) - \frac{9}{8} (16+y^2-8y) \right] dy$$

$$= 9 \int_{y=0}^4 \left[4-y-4-\frac{y^2}{4}+2y \right] dy$$

$$= 9 \int_{y=0}^4 \left(-\frac{y^2}{4} + y \right) dy$$

$$= 9 \left(-\frac{y^3}{12} + \frac{y^2}{2} \right)_0^4$$

$$= 9 \left(-\frac{64}{12} + 8 \right)$$

$$= 9 \left(\frac{8}{3} \right) = 24$$

Q) Evaluate $\int_S \vec{F} \cdot \vec{n} \, ds$ where $\vec{F} = y\vec{i} + 2x\vec{j} - z\vec{k}$ and S is the surface of the plane $2x+y=6$ in the first octant cut off by the plane $z=4$.

G.T

$$\phi = 2x + y - 6$$

$$\nabla \phi = 2\vec{i} + \vec{j}$$

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2\vec{i} + \vec{j}}{\sqrt{4+1}} = \frac{1}{\sqrt{5}} (2\vec{i} + \vec{j})$$

$$\begin{aligned}
 \vec{F} \cdot \vec{n} &= (y\vec{i} + 2x\vec{j} - z\vec{k}) \cdot \frac{1}{\sqrt{5}} (2\vec{i} + \vec{j}) \\
 &= \frac{1}{\sqrt{5}} (2y + 2x)
 \end{aligned}$$

Let R be the projection on xz plane. Then

$$ds = \frac{dx \, dz}{|\vec{n} \cdot \vec{j}|}$$

$$\begin{aligned}
 (\vec{n} \cdot \vec{j}) &= \frac{1}{\sqrt{5}} (2\vec{i} + \vec{j}) \cdot \vec{j} \\
 &= \frac{1}{\sqrt{5}}
 \end{aligned}$$

$$\therefore ds = \frac{dx dz}{1/\sqrt{5}}$$

Limits of x & z :-

On xz plane $y=0$.

$$2x = 6$$

$$x = 3.$$

$$\therefore x \rightarrow 0 \text{ to } 3$$

$$z \rightarrow 0 \text{ to } 4.$$

$$2x + y = 6$$

$$\boxed{y = 6 - 2x} \quad \text{--- (2)}$$

$$\vec{F} \cdot \vec{n} = \frac{1}{\sqrt{5}} (2(6-2x) + 2x)$$

$$= \frac{1}{\sqrt{5}} (12 - 4x + 2x)$$

$$= \frac{1}{\sqrt{5}} (12 - 2x).$$

$$\therefore \int_S \vec{F} \cdot \vec{n} ds = \int_{z=0}^4 \int_{x=0}^3 \frac{1}{\sqrt{5}} (12 - 2x) \frac{dx dz}{1/\sqrt{5}}$$

$$= \int_{z=0}^4 \int_{x=0}^3 (12 - 2x) dx dz$$

$$= \int_{z=0}^4 \left[12x - \frac{2x^2}{2} \right]_0^3 dz$$

$$= \int_{z=0}^4 [36 - 9] dz$$

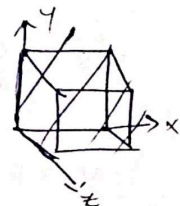
$$= \int_{z=0}^4 27 dz = [27z]_0^4 = [27(4-0)] = 108.$$

$$\frac{27 \times 4}{1 \times 8}$$

Q) Evaluate $\int_S \vec{F} \cdot \vec{n} ds$ where $\vec{F} = x^3 - yz \vec{i} - 2x^2y \vec{j} + 2xz \vec{k}$ and S is the surface of the cube bounded by $x=0, y=0$ & $z=0$ and $x=y=z=a$.

$$\phi = 2x + 3y + 6z - 12$$

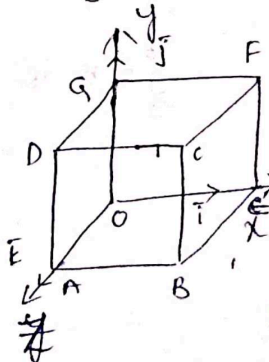
$$\vec{F} \cdot \vec{n} = \frac{1}{7} (3ex - 36 + 18y) \text{--- (1)}$$



In BEFC, $\vec{n} = \vec{i}$, $x=a$.

$$ds = \frac{dy dz}{|\vec{n} \cdot \vec{i}|} = dy dz$$

$y \rightarrow 0 \text{ to } a$ & $z \rightarrow 0 \text{ to } a$



$$\therefore \vec{F} \cdot \vec{n} = x^3 - yz = a^3 - yz$$

$$\therefore \int_{B \in F \in C} \vec{F} \cdot \vec{n} \, ds = \int_{y=0}^a \int_{z=0}^a a^3 - yz \, dy \, dz$$

$$= \int_{y=0}^a \left[a^3 z - y \frac{z^2}{2} \right]_0^a dy$$

$$= \int_0^a \left(a^4 - \frac{a^2}{2} y \right) dy$$

$$= \left[a^4 y - \frac{a^2}{2} \left(\frac{y^2}{2} \right) \right]_0^a$$

$$= a^5 - \frac{a^4}{4} \quad \text{--- (1)}$$

$$\text{In } AOGD, \vec{n} = -\vec{i}, x=0, ds = dy \, dz,$$

$$\vec{F} \cdot \vec{n} = -(x^3 - yz)$$

$$= -(0 - yz) = yz$$

$$\therefore \int_S \vec{F} \cdot \vec{n} \, ds = \int_{y=0}^a \int_{z=0}^a yz \, dy \, dz$$

$$= \left[\frac{y^2}{2} \right]_0^a \left[\frac{z^2}{2} \right]_0^a dy \, dz$$

$$= \frac{a^2}{2} \times \frac{a^2}{2} = \frac{a^4}{4} \quad \text{--- (2)}$$

$$\text{In } CDGF, y=a, \vec{n} = \vec{j}, ds = dx \, dz$$

$$\vec{F} \cdot \vec{n} = -2x^2 y = -2x^2 a. \quad ; \quad x \rightarrow 0 \text{ to } a$$

$$\int_S \vec{F} \cdot \vec{n} \, ds = \int_{x=0}^a \int_{z=0}^a -2x^2 a \, dx \, dz$$

$$= -2a \int_{z=0}^a dz \int_0^a x^2 \, dx$$

$$= -2a \left[z \right]_0^a \left[\frac{x^3}{3} \right]_0^a$$

$$= -2a^2 \times \frac{a^3}{3} = -\frac{2a^5}{3} \quad \text{--- (3)}$$

$$\text{On } ABED, y=0, \vec{n} = -\vec{j}, ds = dx \, dz.$$

$$\vec{F} \cdot \vec{n} = -2x^2 y = 0. (\because y=0)$$

$$\int_{ABED} \vec{F} \cdot \vec{n} \, ds = 0 \quad \text{--- (4)}$$

$$\text{On } ABCD, z=a, \vec{n} = \vec{k}, ds = dx \, dy.$$

$$\vec{F} \cdot \vec{n} = 2$$

$$\int_{ABCD} \vec{F} \cdot \vec{n} \, ds = \int_{x=0}^a \int_{y=0}^a 2 \, dx \, dy$$

$$= 2 \left[x \right]_0^a \left[y \right]_0^a$$

$$= 2a^2 \quad \text{--- (5)}$$

On EDFG, $n = -\bar{k}$, $z = 0$.

$$\bar{F} \cdot \bar{n} = -2.$$

$x \rightarrow 0 \text{ to } a$
 $y \rightarrow 0 \text{ to } a$

$$\int_{\text{EDFG}} \bar{F} \cdot \bar{n} \, ds = \int_{x=0}^a \int_{y=0}^a -2 \, dx \, dy = -2[x]_0^a [y]_0^a$$

$$= -2a^2. \quad \text{--- (6)}$$

$$\therefore \int_S \bar{F} \cdot \bar{n} \, ds = a^5 - \frac{a^4}{4} + \frac{a^4}{4} - \frac{2a^5}{3} + 0 + 2a^2 - 2a^2$$

$$= a^5 - \frac{2a^5}{3} = \frac{a^5}{3} //$$

VOLUME INTEGRAL:-

Suppose $\bar{F}$ is a vector point function bounded by the region R with volume V then $\int \bar{F} \, dv$ is called volume integral, where

$$dv = dx \, dy \, dz \text{ and } \bar{F} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}.$$

$$\int_V \bar{F} \, dv = \iiint_R (f_1 \, dx \, dy \, dz) \bar{i} + \iiint_R (f_2 \, dx \, dy \, dz) \bar{j}$$

$$+ \iiint_R (f_3 \, dx \, dy \, dz) \bar{k}.$$

6) If $\bar{F} = 2xz \bar{i} - x^2 \bar{j} + y^2 \bar{k}$. Evaluate $\int \bar{F} \, dv$ where V is the region bounded by the surfaces $x=0$, $x=2$, $y=0$, $y=6$, $z=x^2$, $z=4$.

G.T

$$x \rightarrow 0 \text{ to } 2$$

$$y \rightarrow 0 \text{ to } 6$$

$$z \rightarrow x^2 \text{ to } 4$$

$$\bar{F} = 2xz \bar{i} - x^2 \bar{j} + y^2 \bar{k} \therefore \int \bar{F} \, dv = \iiint_V [2xz \bar{i} - x^2 \bar{j} + y^2 \bar{k}] \, dx \, dy \, dz$$

$$= \bar{i} \int_{x=0}^2 2x \int_{y=0}^6 \left[\int_{z=x^2}^4 z \, dz \right] dy \, dx - \bar{j} \int_{x=0}^2 \left[x \, dx \int_{y=0}^6 \int_{z=x^2}^4 dz \right]$$

$$+ \bar{k} \int_{x=0}^2 dx \left[\int_{y=0}^6 y^2 \, dy \int_{z=x^2}^4 dz \right]$$

$$= \bar{i} \int_{x=0}^2 2x [y]_0^6 \left[\frac{z^2}{2} \right]_{x^2}^4 dx - \bar{j} \int_{x=0}^2 x \, dx [y]_0^6 [z]_{x^2}^4 +$$

$$\bar{k} \int_{x=0}^2 dx \left(\frac{y^3}{3} \right)_0^6 [z]_{x^2}^4.$$

$$= 6\bar{i} \int_{x=0}^2 (16x - x^5) \, dx - 6\bar{j} \int_{x=0}^2 x(4 - x^2) \, dx + \frac{216}{3} \int_{x=0}^2 (4 - x^2) \, dx$$

$$= 6\bar{i} \left[\frac{16x^2}{2} - \frac{x^6}{6} \right]_0^2 - 6\bar{j} \left[\frac{4x^2}{2} - \frac{x^3}{3} \right]_0^2 + 72 \left[4x - \frac{x^3}{3} \right]_0^2$$

$$= 6\bar{i} \left(32 - \frac{32}{3} \right) - 6\bar{j} (8 - 4) + 72\bar{k} \left(8 - \frac{8}{3} \right)$$

$$= 198\bar{i} - 24\bar{j} + (24 \times 16)\bar{k}$$

$$= 198\bar{i} - 24\bar{j} + 384\bar{k}$$

2) Evaluate $\int_V \phi \, dv$ where $\phi = 45x^2y$ and closed region bounded by planes $4x+2y+z=8$, $x=0$, $y=0$, $z=0$.

Sol G.T

$$4x+2y+z=8 \quad \text{--- ①}$$

$$z = 8 - 4x - 2y \Rightarrow \boxed{z=0 \text{ to } 8-4x-2y}$$

Put $z=0$ in ①

$$4x+2y=8$$

$$2y = 8 - 4x$$

$$y = 4 - 2x \Rightarrow \boxed{y \rightarrow 0 \text{ to } 4-2x}$$

Put $y=0$ in ②

$$4x=8 \Rightarrow x=2 \Rightarrow \boxed{x \rightarrow 0 \text{ to } 2}$$

$$\iiint_V \phi \, dv = \iiint_V 45x^2y \, dx \, dy \, dz$$

$$= 45 \int_{x=0}^2 \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} x^2y \, dz \, dy \, dx$$

$$= 45 \int_{x=0}^2 \int_{y=0}^{4-2x} x^2y [z]_0^{8-4x-2y} dy \, dx$$

$$= 45 \int_{x=0}^2 \int_{y=0}^{4-2x} x^2y [8-4x-2y] dy \, dx$$

$$= 45 \int_{x=0}^2 \int_{y=0}^{4-2x} 8x^2y - 4x^3y - 2x^2y^2 dy \, dx$$

$$= 45 \int_{x=0}^2 \left[8x^2 \frac{y^2}{2} - 4x^3 \frac{y^2}{2} - 2x^2 \frac{y^3}{3} \right]_{y=0}^{4-2x} dx$$

$$= 45 \int_{x=0}^2 \left[8x^2 \frac{(4-2x)^2}{2} - 4x^3 \frac{(4-2x)^2}{2} - 2x^2 \frac{(4-2x)^3}{3} \right] dx$$

$$= 45 \int_{x=0}^2 \left[4x^2(4-2x)^2 - 2x^3(4-2x)^2 - \frac{2}{3}x^2(4-2x)^3 \right] dx$$

$$= 45 \int_{x=0}^2 (4-2x)^2 \left[4x^2 - 2x^3 - \frac{2}{3}x^2(4-2x) \right] dx$$

$$= 45 \int_{x=0}^2 (4-2x)^2 \left[4x^2 - 2x^3 - \frac{8}{3}x^2 + \frac{4}{3}x^3 \right] dx$$

$$= 45 \int_{x=0}^2 (4-2x)^2 \left[\frac{4}{3}x^2 - \frac{2}{3}x^3 \right] dx$$

$$= \frac{15}{1} \int_{x=0}^2 4(2-x)^2 \frac{2}{3} (2x^2 - x^3) dx = 120 \int_{x=0}^2 (2-x)^2 (2x^2 - x^3) dx$$

$$120 \int_{x=0}^2 (4+x^2-4x)(2x^2-x^3) dx$$

$$= 120 \int_{x=0}^2 8x^2 + 2x^4 - 8x^3 - 4x^3 - x^5 + 4x^4 dx$$

$$= 120 \int_{x=0}^2 -x^5 + 6x^4 - 12x^3 + 8x^2 dx.$$

$$= 120 \left[-\frac{x^6}{6} + 6\left(\frac{x^5}{5}\right) - 12\left(\frac{x^4}{4}\right) + 8\left(\frac{x^3}{3}\right) \right]_0^2$$

$$= 120 \left[-\frac{64}{6} + 6\left(\frac{32}{5}\right) - 12\left(\frac{16}{4}\right) + 8\left(\frac{8}{3}\right) \right]$$

$$= 120 \left[-\frac{64}{6} + \frac{192}{5} - 48 + \frac{64}{3} \right]$$

$$= 120 \left[-\frac{64}{6} + \frac{192}{5} - \frac{32}{3} \right] = 48$$

$$= -32$$

$$= 120 \left[-\frac{64}{6} + \frac{192}{5} - 48 + \frac{64}{3} \right]$$

$$= \left(\frac{16}{15}\right) 120 = 128 //$$

VECTOR INTEGRAL THEOREMS:-

Green's theorem:- If 'R' is a closed region in xy plane bounded by a simple closed curve 'C' and if M, N are continuous functions of x & y having continuous derivatives in R. then

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \text{ where } C \text{ is traversed in the positive direction.}$$

Q) Using green's theorem evaluate

$\int_C (2xy - x^2) dx + (x^2 + y^2) dy$ where C is closed curve of region $x^2 = y, y^2 = x$

Given curves

$$x^2 = y, y^2 = x$$

$$y = y^4$$

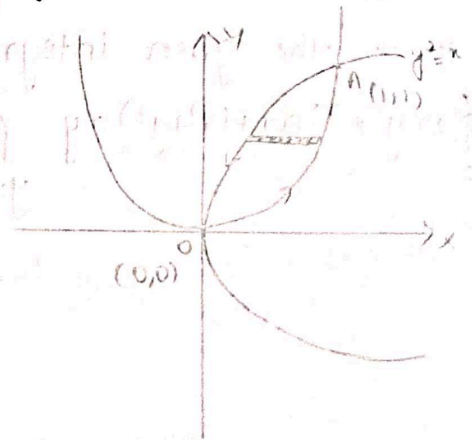
$$y(1 - y^3) = 0$$

$$y = 0, 1$$

$$y = 0 \Rightarrow x = 0$$

$$y = 1 \Rightarrow x = 1$$

$\therefore$ Points of intersection are $(0,0)$ & $(1,1)$



If R is a region of a simple closed curve C according to the green's theorem it states that $\oint_C Mdx + Ndy = \iint_R \left[\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \right] dx dy$

From the given integral we have

$$M = 2xy - x^2 \quad N = x^2 + y^2 \quad \text{Now,}$$

$$\frac{\partial M}{\partial y} = 2x; \quad \frac{\partial N}{\partial x} = 2x$$

To find the limits consider a horizontal strip (or a vertical strip) PQ then

limits:-

$$x \rightarrow y^2 \text{ to } \sqrt{y} \quad (\text{along } PQ)$$

$$y \rightarrow 0 \text{ to } 1 \quad (\text{from } OA)$$

Hence the given integral can be written as

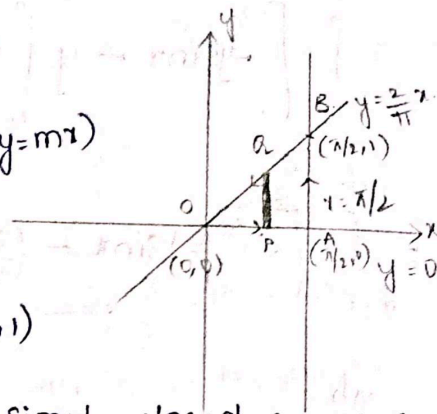
$$\oint_C (2xy - x^2) dx + (x^2 + y^2) dy = \int_{y=0}^1 \left[\int_{x=y^2}^{\sqrt{y}} (2x - 2x) dx \right] dy = 0$$

a) Evaluate by green's theorem $\oint_C (y - \sin x) dx + \cos x dy$ where C is the triangle enclosed by the lines $y=0$, $x=\pi/2$, $\pi y=2x$.

Given curves

$$y=0, \quad x=\pi/2, \quad \pi y=2x$$

$$y = \frac{2}{\pi} x \quad (y=mx)$$



Points of intersection

$$\text{are } (0,0), (\pi/2, 0), (\pi/2, 1)$$

If R is a region of a simple closed curve C according to the green's theorem it states that $\oint_C Mdx + Ndy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy$.

From the given integral we have

$$M = y - \sin x, \quad N = \cos x$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -\sin x$$

To find the limits consider a vertical strip PQ in the $\Delta^k OAB$ then

limits

$$y \rightarrow 0 \text{ to } \frac{2}{\pi} x \quad (\text{from } OA \text{ to } AB)$$

$$x \rightarrow 0 \text{ to } \pi/2 \quad (\text{from left most point to right most point})$$

Hence the given integral can be written as

$$\begin{aligned} \oint_C (y \sin x) dx + \cos x dy &= \int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2}{\pi}x} [-\sin x - 1] dy dx \\ &= \int_{x=0}^{\pi/2} \left[-y \sin x - y \right]_0^{\frac{2}{\pi}x} dx \\ &= \int_{x=0}^{\pi/2} \left[-\left(\frac{2x}{\pi}\right) \sin x - \left(\frac{2}{\pi}x\right) \right] dx \\ &= \int_{x=0}^{\pi/2} - \left(\frac{2x}{\pi} \sin x + \frac{2}{\pi}x \right) dx \\ &= -\frac{2}{\pi} \int_{x=0}^{\pi/2} (x \sin x + x) dx \\ &= -\frac{2}{\pi} \left[x(-\cos x) + \sin x + \frac{x^2}{2} \right]_0^{\pi/2} \\ &= -\frac{2}{\pi} \left[\left(\frac{\pi}{2}(-\cos(\pi/2)) + \sin(\pi/2) + \frac{(\pi/2)^2}{2} \right) - \left(0(-\cos 0^\circ) + \sin 0 + 0 \right) \right] \end{aligned}$$

$$= -\frac{2}{\pi} \left[1 + \frac{\pi^2}{4} \times \frac{1}{2} \right]$$

$$\left(-\frac{\pi}{4} + \frac{2}{\pi} \right)$$

$$= -\frac{2}{\pi} \left[1 + \frac{\pi^2}{8} \right]$$

$$= -\frac{2}{\pi} - \frac{\pi}{4} = -2 \left(\frac{\pi}{4} + \frac{2}{\pi} \right)$$

4) ~~Verify Green's theorem for $\oint_C (2x^2 - y^2) dx + (x^2 + y^2) dy$ where C is the boundary of the~~

4) Verify Green's theorem for $\oint_C (xy + y^2) dx + x^2 dy$ where C is bounded by $y = x^2$, $y = x$.

Given curves are
 $y = x^2$, $y = x$.

Sub $y = x$ in $y = x^2$.

$$x = x^2$$

$$x^2 - x = 0$$

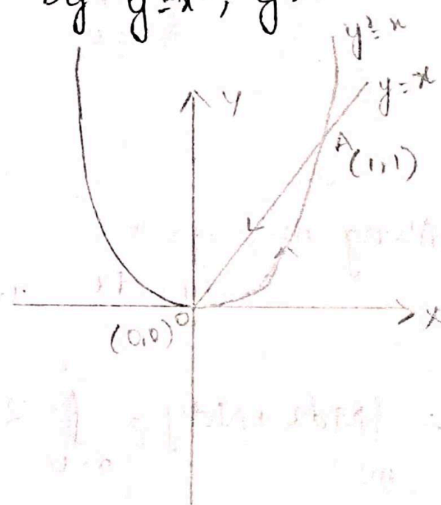
$$x(x-1) = 0$$

$$x = 0, x = 1$$

$$\Rightarrow x = 0, y = 0$$

$$\Rightarrow x = 1, y = 1$$

$\therefore$ Points of Intersection are $(1,1)$, $(0,0)$



$$\oint_C Mdx + Ndy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy \quad \text{--- (1)}$$

L.H.S

$$\oint_C Mdx + Ndy = \int_{OA} Mdx + Ndy + \int_{AO} Mdx + Ndy \quad \text{--- (2)}$$

Along OA, $y = x^2$
 $dy = 2x dx$ & $x \rightarrow 0$ to 1

$$\begin{aligned} \therefore \int_{OA} Mdx + Ndy &= \int_{x=0}^1 (x^3 + x^4) dx + \int_0^1 2x^3 dx \\ &= \int_{x=0}^1 (x^4 + 3x^3) dx = \left[\frac{x^5}{5} + 3\left(\frac{x^4}{4}\right) \right]_0^1 \\ &= \frac{1}{5} + \frac{3}{4} = \frac{19}{20} \end{aligned}$$

Along AO, $y = x$
 $dy = dx$ & $x \rightarrow 0$ to 1

$$\begin{aligned} \therefore \int_{AO} Mdx + Ndy &= \int_{x=0}^1 2x^2 dx + x^2 dx \\ &= \int_{x=1}^0 3x^2 dx = 3 \left[\frac{x^3}{3} \right]_0^1 \\ &= 3(0 - 1/3) = -1 \end{aligned}$$

$$\oint_C Mdx + Ndy = \frac{19}{20} - 1 = -1/20$$

RHS

G.T. $M = xy + y^2 \Rightarrow \frac{\partial M}{\partial y} = x + 2y$

$N = x^2 \Rightarrow \frac{\partial N}{\partial x} = 2x$

Limits:- $x \rightarrow y$ to $\sqrt{y}$
 $y \rightarrow 0$ to 1

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \int_{y=0}^1 \left[\int_{x=y}^{\sqrt{y}} (2x - x - 2y) dx \right] dy$$

$$= \int_{y=0}^1 \left[\int_{x=y}^{\sqrt{y}} (x - 2y) dx \right] dy$$

$$= \int_{y=0}^1 \left[\frac{x^2}{2} - 2yx \right]_y^{\sqrt{y}} dy$$

$$= \int_{y=0}^1 \left[\frac{y}{2} - 2y^{3/2} - \left(\frac{y^2}{2} - 2y^2 \right) \right] dy$$

$$= \int_{y=0}^1 \left[\frac{y}{2} - 2y^{3/2} + \frac{3y^2}{2} \right] dy$$

$$= \left[\frac{y^2}{4} - \frac{4}{5} y^{5/2} + \frac{y^3}{2} \right]_0^1$$

$$= \frac{1}{4} - \frac{4}{5} + \frac{1}{2}$$

$$= -1/20$$

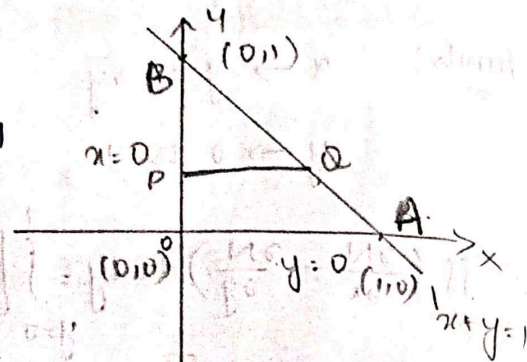
$\therefore$ LHS = RHS
 Hence verified

5) Verify Green's theorem for $\oint (3x^2 - 8y^2) dx + (4y - 6xy) dy$ where C is the region bounded by $x=0$, $y=0$, $x+y=1$.

Given curves are

$$x=0, y=0, x+y=1$$

x	0	1
y	1	0



Points of intersection are $(0,0)$, $(1,0)$, $(0,1)$

$$\oint_C M dx + N dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy \quad \text{--- (1)}$$

L.H.S:-

$$\oint_C M dx + N dy = \int_{OA} M dx + N dy + \int_{AB} M dx + N dy + \int_{BO} M dx + N dy$$

$$\int_{OB} M dx + N dy$$

Along OA, $y=0$
 $dy=0$

$x \rightarrow 0$ to 1 .

$$\therefore \oint_{OA} M dx + N dy = \int_0^1 (3x^2) dx = \left[\frac{3x^3}{3} \right]_0^1 = (1-0) = 1$$

Along AB, $x+y=1$

$$x=1-y$$

$$dx = -dy$$

$y \rightarrow 0$ to 1 .

$$\therefore \oint_{AB} M dx + N dy = \int_0^1 (3(1-y)^2 - 8y^2)(-dy) + (4y - 6(1-y)y) dy$$

$$= \int_0^1 [8y^2 - 3(1-y)^2 + 4y - 6y + 6y^2] dy$$

$$= \int_0^1 [14y^2 - 3(1-y)^2 - 2y] dy$$

$$= \left[14 \left[\frac{y^3}{3} \right] - 3 \left[\frac{(1-y)^3}{-3} \right] - y^2 \right]_0^1$$

$$= \frac{14}{3} + 0 - 1 - (0 - 1 - 0)$$

$$= \frac{14}{3} - 2 = \frac{8}{3}$$

Along BO, $x=0$
 $dx=0$

$y \rightarrow 1$ to 0 .

$$\begin{aligned}
 \therefore \int M dx + N dy &= \int_{y=1}^0 (-8y^2) + 4y dy \\
 &= \int_{y=1}^0 -8y^2 + 4y dy \\
 &= -8 \left[\frac{y^3}{3} \right]_1^0 + 2 \left[\frac{y^2}{1} \right]_1^0 \\
 &= -8 \left[0 - \frac{1}{3} \right] + 2[0 - 1] \\
 &= -\frac{8}{3} - 2 \\
 &= -\frac{14}{3}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \int M dx + N dy &= \int_{y=1}^0 4y dy \\
 &= 4 \left[\frac{y^2}{2} \right]_1^0 \\
 &= 2[0 - 1] \\
 &= -2
 \end{aligned}$$

$$\begin{aligned}
 \therefore \oint M dx + N dy &= 1 + \frac{8}{3} - 2 \\
 &= \frac{8}{3} - 1 = \frac{5}{3}
 \end{aligned}$$

RHS:-

$$\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy \quad \begin{aligned} M &= 3x^2 - 8y^2 \\ N &= 4y - 6xy \end{aligned}$$

$$\frac{\partial M}{\partial y} = -16y$$

$$\frac{\partial N}{\partial x} = -6y$$

To find the limits consider a horizontal strip PQ then

limits:-

$$x \rightarrow 0 \text{ to } 1-y$$

$$y \rightarrow 0 \text{ to } 1$$

$$\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy = \iint_R [-6y - (-16y)] dx dy$$

$$= \int_{y=0}^1 \left[\int_{x=0}^{1-y} (10y) dx \right] dy$$

$$= \int_{y=0}^1 \left[10y [x]_0^{1-y} \right] dy$$

$$= \int_{y=0}^1 [10y(1-y)] dy$$

$$= \int_{y=0}^1 (10y - 10y^2) dy$$

$$= 5 \left[\frac{y^2}{2} \right]_0^1 - 10 \left[\frac{y^3}{3} \right]_0^1$$

$$= 5(1-0) - 10\left(\frac{1}{3}-0\right)$$

$$= 5 - \frac{10}{3}$$

$$= \frac{5}{3} //$$

6) Use Green's theorem evaluate $\oint_C x^2(1+y)dx + (x^3+y^3)dy$ where C is the square bounded by $y = \pm 1$ and $x = \pm 1$

Given lines are

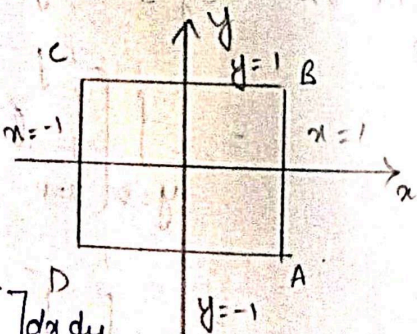
$$y = \pm 1, x = \pm 1$$

By Green's theorem

$$\oint_C M dx + N dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy$$

From the given integral we have

$$M = x^2(1+y) \quad N = x^3 + y^3$$



$$\frac{\partial M}{\partial y} = x^2$$

$$\frac{\partial N}{\partial x} = 3x^2$$

Limits :-

$$x \rightarrow -1 \text{ to } +1$$

$$y \rightarrow -1 \text{ to } +1$$

$$\int_{x=-1}^1 \int_{y=-1}^1 (3x^2 - x^2) dx dy$$

$$\int_{x=-1}^1 \int_{y=-1}^1 (2x^2) dx dy$$

$$= \int_{x=-1}^1 [2x^2 [y]_{-1}^1] dx$$

$$= \int_{x=-1}^1 [2x^2(1+1)] dx$$

$$= \int_{x=-1}^1 4x^2 dx$$

$$= 4 \left[\frac{x^3}{3} \right]_{-1}^1 = 4 \left[\frac{1}{3} - \left(-\frac{1}{3} \right) \right]$$

$$= 4 \left[\frac{2}{3} \right] = 8/3$$

7) Find $\oint_C (x^2+y^2)dx + (3xy^2)dy$ where C is the circle $x^2+y^2=4$ in xy plane.

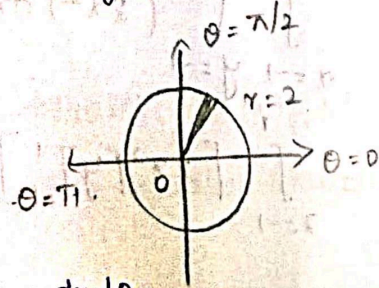
Sol $\oint_C (x^2+y^2)dx + (3xy^2)dy$ is of the form $\oint_C Mdx + Ndy$

From Green's theorem,

$$\oint_C Mdx + Ndy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy.$$

$$M = x^2 + y^2 \quad \left| \quad N = 3xy^2 \right.$$

$$\frac{\partial M}{\partial y} = 2y \quad \left| \quad \frac{\partial N}{\partial x} = 3y^2 \right.$$



$$x = r \cos \theta, \quad y = r \sin \theta, \quad dx dy = r dr d\theta.$$

$$\therefore x^2 + y^2 = 4 \Rightarrow r^2 \cos^2 \theta + r^2 \sin^2 \theta = 4$$

$$r^2 = 4$$

$$\boxed{r = 2}$$

$$\therefore r \rightarrow 0 \text{ to } 2 \text{ \& } \theta \rightarrow 0 \text{ to } 2\pi.$$

$$\therefore \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy = \int_0^{2\pi} \int_0^2 (3y^2 - 2y) dx dy$$

$$= \int_0^{2\pi} \int_0^2 [3(r \sin \theta)^2 - 2(r \sin \theta)] r dr d\theta$$

$$= \int_0^{2\pi} \int_0^2 (3r^2 \sin^2 \theta - 2r \sin \theta) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^2 (3r^3 \sin^2 \theta - 2r^2 \sin \theta) dr d\theta$$

$$= \int_0^{2\pi} \left[3 \sin^2 \theta \int_0^2 r^3 dr - 2 \sin \theta \int_0^2 r^2 dr \right] d\theta$$

$$= \int_0^{2\pi} \left[3 \sin^2 \theta \left[\frac{r^4}{4} \right]_0^2 - 2 \sin \theta \left[\frac{r^3}{3} \right]_0^2 \right] d\theta$$

$$= \int_0^{2\pi} \left[3 \sin^2 \theta [4] - \frac{16 \sin \theta}{3} \right] d\theta$$

$$= 12 \int_0^{2\pi} \sin^2 \theta d\theta - \frac{16}{3} \int_0^{2\pi} \sin \theta d\theta$$

$$= 12 \int_0^{2\pi} \frac{1 - \cos 2\theta}{2} d\theta - \frac{16}{3} \int_0^{2\pi} \sin \theta d\theta$$

$$= 12 \left[\int_0^{2\pi} \frac{1}{2} d\theta - \int_0^{2\pi} \frac{\cos 2\theta}{2} d\theta \right] - \frac{16}{3} \int_0^{2\pi} \sin \theta d\theta$$

$$= 12 \left[\frac{1}{2} (\theta)_0^{2\pi} - \frac{1}{2} \left[\frac{\sin 2\theta}{2} \right]_0^{2\pi} \right] + \frac{16}{3} [\cos \theta]_0^{2\pi}$$

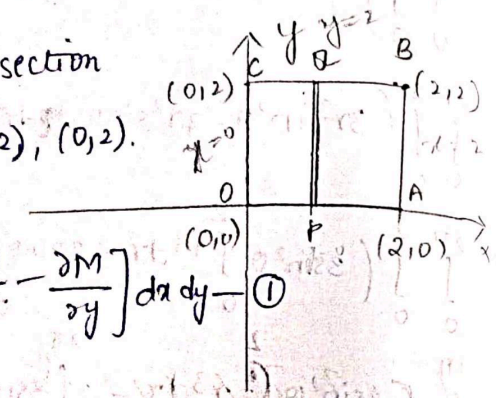
$$= 12 \left[\frac{1}{2} [2\pi - 0] - \frac{1}{2} \left[\frac{\sin(2\pi)}{2} - 0 \right] \right] + \frac{16}{3} [\cos 2\pi]$$

$$= 12 [\pi - 0] + \frac{16}{3} \times 1 = 12\pi + \frac{16}{3}$$

$$= 12\pi + \frac{16}{3}$$

8) Verify Green's theorem in the plane for $\oint_C (x^2 - xy^3) dx + (y^2 - 2xy) dy$ where C is the square with vertices $(0,0), (2,0), (2,2), (0,2)$.

Given points of intersection are $(0,0), (2,0), (2,2), (0,2)$.



$$\oint_C M dx + N dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy \quad \text{--- (1)}$$

LHS:-

$$\oint_C M dx + N dy = \int_{OA} M dx + N dy + \int_{AB} M dx + N dy + \int_{BC} M dx + N dy + \int_{CO} M dx + N dy$$

Along OA, $y=0$
 $dy=0$ $x \rightarrow 0$ to 2

$$\int_{OA} M dx + N dy = \int_{x=0}^2 x^2 dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3}$$

Along AB, $x=2$
 $dx=0$ $y \rightarrow 0$ to 2 .

$$\begin{aligned} \int_{AB} M dx + N dy &= \int_{y=0}^2 (4 - 2y^3)(0) + (y^2 - 2(2)y) dy \\ &= \int_{y=0}^2 y^2 - 4y dy = \left[\frac{y^3}{3} - 2y^2 \right]_0^2 = \frac{8}{3} - 8 = -\frac{16}{3} \end{aligned}$$

$$= \left(\frac{8}{3} - 0 \right) - 2(4 - 0) = \frac{8}{3} - 8 = -\frac{16}{3}$$

Along BC, $y=2$
 $dy=0$
 $x \rightarrow 2$ to 0 .

$$\begin{aligned} \int_{BC} M dx + N dy &= \int_{x=2}^0 (x^2 - x(2)^3) dx \\ &= \int_{x=2}^0 (x^2 - 8x) dx = \left[\frac{x^3}{3} - 4x^2 \right]_2^0 \\ &= (0 - 0) - \left(\frac{8}{3} - 16 \right) = -\frac{8}{3} + 16 = \frac{40}{3} \end{aligned}$$

Along CO, $x=0$
 $dx=0$ $y \rightarrow 2$ to 0 .

$$\begin{aligned} \int_{CO} M dx + N dy &= \int_{y=2}^0 y^2 dy = \left[\frac{y^3}{3} \right]_2^0 \\ &= (0 - \frac{8}{3}) = -\frac{8}{3} \end{aligned}$$

$$\begin{aligned} \oint_C M dx + N dy &= \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} \\ &= \frac{24}{3} = 8 \end{aligned}$$

RHS:-

$$\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy$$

$$M = x^2 - 2y^3$$

$$N = y^2 - 2xy$$

$$\frac{\partial M}{\partial y} = -3y^2$$

$$\frac{\partial N}{\partial x} = -2y$$

To find the limits consider a horizontal (or vertical strip) & then limits are

$$\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy$$

$$y \rightarrow 0 \text{ to } 2$$

$$x \rightarrow 0 \text{ to } 2$$

$$\int_{x=0}^2 \int_{y=0}^2 [-2y + 3y^2] dx dy$$

$$= \int_{x=0}^2 \left[2 \left[\frac{y^2}{2} \right] + 3x \left[\frac{y^3}{3} \right] \right]_0^2 dx$$

$$= \int_{x=0}^2 [-4 + 8x] dx$$

$$= \int_{x=0}^2 -4 + 8x dx = -4 \left[x \right]_0^2 + 8 \left[\frac{x^2}{2} \right]_0^2$$

$$= -4[2-0] + 4[4-0]$$

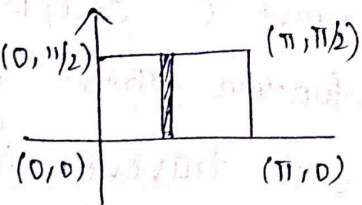
$$= -8 + 16 = 8$$

$$LHS = RHS$$

Hence Green's theorem is verified.

a) Evaluate by Green's theorem $\oint e^{-x} \sin y dx + e^{-x} \cos y dy$ where C is the rectangle whose vertices are $(0,0)$, $(\pi,0)$, $(\pi, \pi/2)$, $(0, \pi/2)$.

$$\oint M dx + N dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy$$



Consider vertical strip

$$y \text{ limits} \rightarrow 0 \text{ to } \pi/2$$

$$x \text{ limits} \rightarrow 0 \text{ to } \pi \Rightarrow \int_0^{\pi} \int_0^{\pi/2} e^{-x} \sin y dx + e^{-x} \cos y dy$$

$$M = e^{-x} \sin y \quad N = e^{-x} \cos y$$

$$\frac{\partial M}{\partial y} = e^{-x} \cos y \quad \frac{\partial N}{\partial x} = -\cos y e^{-x}$$

$$= \int_0^{\pi} \int_0^{\pi/2} [-\cos y e^{-x} - e^{-x} \cos y] dx dy$$

$$= \int_0^{\pi/2} \left[\int_0^{\pi} (-\cos y e^{-x} - e^{-x} \cos y) dx \right] dy$$

$$= \int_0^{\pi/2} \left[-\cos y \int_0^{\pi} e^{-x} dx - \cos y \int_0^{\pi} e^{-x} dx \right] dy$$

$$= \int_0^{\pi/2} \left[-\cos y \left[\frac{e^{-x}}{-1} \right]_0^{\pi} - \cos y \left[\frac{e^{-x}}{-1} \right]_0^{\pi} \right] dy$$

$$= \int_0^{\pi/2} \left[-\cos y \left[\frac{e^{-\pi}}{-1} - \frac{e^{-0}}{-1} \right] - \cos y \left[\frac{e^{-\pi}}{-1} - \frac{e^{-0}}{-1} \right] \right] dy$$

$$= \int_0^{\pi/2} \left[\cos y (e^{-\pi} - 1) + \cos y (e^{-\pi} - 1) \right] dy$$

$$= \int_0^{\pi/2} 2 \cos y [e^{-\pi} - 1] dy = 2 [e^{-\pi} - 1] \int_0^{\pi/2} \cos y dy = 2 [e^{-\pi} - 1] [\sin y]_0^{\pi/2}$$

$$= 2 [e^{-\pi} - 1] [1 - 0] = 2 [e^{-\pi} - 1]$$

STOKE'S THEOREM:- Let 'S' be an open surface bounded by a closed non intersecting curve 'C' and if $\vec{F}$ is differentiable vector point function then $\oint_C \vec{F} \cdot d\vec{r} = \int_S \text{curl } \vec{F} \cdot \vec{n} \, ds$ where C is traversed in the positive direction and $\vec{n}$ is outward drawn unit normal at any point of the surface.

Problems:-

1) If $\vec{F} = y\vec{i} + (x - 2xz)\vec{j} - xy\vec{k}$ evaluate $\int_S \text{curl } \vec{F} \cdot \vec{n} \, ds$ where S is surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane.

2) Verify Stoke's theorem for $\vec{F} = (2x - y)\vec{i} - yz^2\vec{j} + y^2z\vec{k}$ over the upper half of the sphere $x^2 + y^2 + z^2 = 1$ bounded by the projection of xy plane.

3) Verify Stoke's theorem for the function $\vec{F} = x^2\vec{i} + xy\vec{j}$ integrated round the square in the plane $z=0$ along the lines $x=0, y=0, x=a, y=a$.

4) Verify Stoke's theorem for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ and surface S of the part of the sphere $x^2 + y^2 + z^2 = 1$ above xy plane.

5)

$$6. \vec{F} = y\vec{i} + (x - 2xz)\vec{j} - xy\vec{k}$$

In xy plane, $z=0$.

$$\vec{F} \cdot d\vec{r} = ydx + xdy$$

$$x^2 + y^2 + z^2 = a^2$$

$$\Rightarrow \boxed{x^2 + y^2 = a^2}$$

$$x = a \cos \theta, y = a \sin \theta$$

$$dx = -a \sin \theta \, d\theta, dy = a \cos \theta \, d\theta, \theta \rightarrow 0 \text{ to } 2\pi$$

$$\begin{aligned} \therefore \oint_C \vec{F} \cdot d\vec{r} &= \int_S \text{curl } \vec{F} \cdot \vec{n} \, ds \\ &= \int_0^{2\pi} [a \sin \theta (-a \sin \theta) + (a \cos \theta)(a \cos \theta)] d\theta \\ &= \int_0^{2\pi} (-a^2 \sin^2 \theta + a^2 \cos^2 \theta) d\theta \\ &= \int_0^{2\pi} a^2 (\cos^2 \theta - \sin^2 \theta) d\theta \\ &= a^2 \int_0^{2\pi} \cos 2\theta \, d\theta \\ &= a^2 \left[\frac{\sin 2\theta}{2} \right]_0^{2\pi} \\ &= a^2 [0] = 0 \end{aligned}$$

2) Verify Stokes's theorem for $\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ over the upper half of surface of sphere $x^2 + y^2 + z^2 = 1$ bounded by projection of xy plane.

LHS:-

$$\oint_C \vec{F} \cdot d\vec{r} = \int_C F_1 dx + F_2 dy + F_3 dz$$

$$= \int_C (2x-y)dx + (-yz^2)dy - (y^2z)dz$$

In xy plane $z=0$

$$= \int_C (2x-y)dx$$

$$\text{Let } x = r \cos \theta, y = r \sin \theta.$$

$$dx = -r \sin \theta d\theta$$

$$= \int_0^{2\pi} (2(r \cos \theta) - r \sin \theta)(-r \sin \theta d\theta)$$

$$= - \int_0^{2\pi} [2r^2 \cos \theta \sin \theta - r^2 \sin^2 \theta] d\theta$$

$$= - \int_0^{2\pi} (r^2 \sin 2\theta - r^2 \sin^2 \theta) d\theta$$

$$= - \left[\int_0^{2\pi} (\sin 2\theta - \sin^2 \theta) d\theta \right]$$

$$\begin{aligned}
 &= \int_0^{2\pi} \sin^2 \theta d\theta - \int_0^{2\pi} \sin 2\theta d\theta \\
 &= \int_0^{2\pi} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta + \left[\frac{\cos 2\theta}{2} \right]_0^{2\pi}
 \end{aligned}$$

$$= \frac{1}{2} \left[\theta \right]_0^{2\pi} - \frac{\sin 2\theta}{2} \Big|_0^{2\pi} + \frac{1}{2} - \frac{1}{2}$$

$$= \frac{1}{2} [2\pi - 0]$$

$$= \pi - \textcircled{1}$$

RHS:-

$\nabla \times \mathbf{F}$ is as follows

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial}{\partial y} (-y^2z) + \frac{\partial}{\partial z} (yz^2) \right) - \hat{j} \left(\frac{\partial}{\partial x} (-y^2z) - \frac{\partial}{\partial z} (2xy) \right)$$

$$+ \hat{k} \left(\frac{\partial}{\partial x} (-yz^2) - \frac{\partial}{\partial y} (2x-y) \right)$$

$$= \hat{i} (-2yz + 2yz) - \hat{j} (0-0) + \hat{k} (0-(-2))$$

$$= 2\hat{k}$$

$$d\vec{s} = \frac{dx dy}{|\vec{n} \cdot \vec{k}|}$$

$$\phi = x^2 + y^2 + z^2$$

$$\nabla \phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla \phi = 2(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{\sqrt{4}}$$

$$= x\vec{i} + y\vec{j} + z\vec{k}$$

$$\boxed{\vec{n} \cdot \vec{k} = z}$$

$$\int_S \text{curl } \vec{F} \cdot \vec{n} \, ds = \int dx dy$$

$$= \int_0^{2\pi} \int_0^1 r dr d\theta = \pi \quad \text{--- (2)}$$

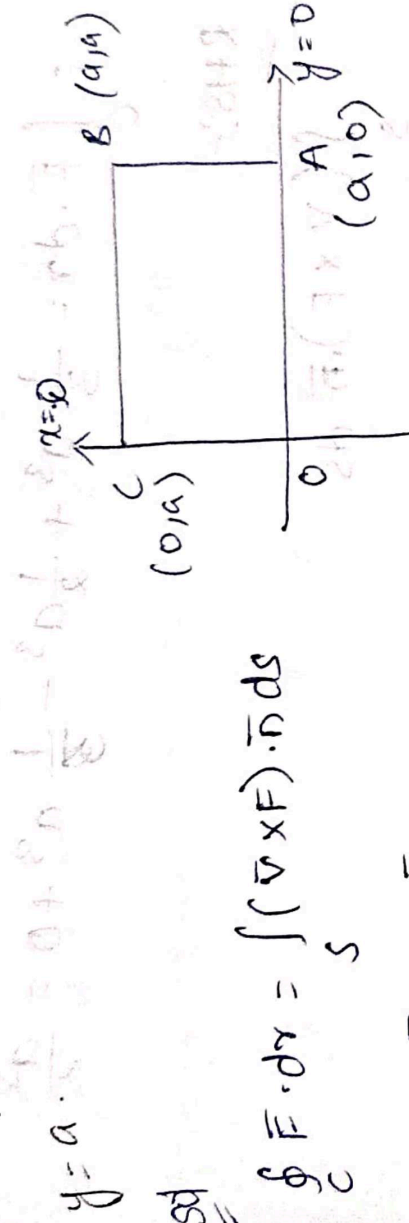
$$LHS = RHS$$

$\therefore$ Stoke's theorem verified

3) Verify Stokes's theorem for the function

$\vec{F} = x^2\vec{i} + xy\vec{j}$, integrated round the square in

the plane $z=0$, along the lines $x=0, y=0, x=a, y=a$.



$$\oint_C \vec{F} \cdot d\vec{r} = \int_S (\nabla \times \vec{F}) \cdot \vec{n} \, ds$$

$$\vec{F} = x^2\vec{i} + xy\vec{j}$$

$$= \oint_C x^2 dx + xy dy$$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_A \vec{F} \cdot d\vec{r} + \int_B \vec{F} \cdot d\vec{r} + \int_C \vec{F} \cdot d\vec{r} + \int_D \vec{F} \cdot d\vec{r}$$

Along OA ($y=0, z=0, dy=0$)

$$\int_0^a x^2 dx = \left[\frac{x^3}{3} \right]_0^a = \frac{a^3}{3} \quad \text{--- (1)}$$

Along AB ($x=a, z=0, dx=0, dz=0$)

$$\int_A^B a y dy$$

$$a \int_0^a y dy \Rightarrow a \left[\frac{y^2}{2} \right]_0^a = a \left[\frac{a^2}{2} \right] = \frac{a^3}{2} \quad \text{--- (2)}$$

Along BC ($y=a, z=0, dy=0, dz=0$)

$$\int_a^0 x^2 dx = \left[\frac{x^3}{3} \right]_a^0 = -\frac{a^3}{3} \quad \text{--- (3)}$$

along CO ($x=0, y=a, dx=0, dy=0$)

$$\oint_C 0 \times 0 + 0 \times a = 0 \quad \text{--- (4)}$$

$$= \int_C \vec{F} \cdot d\vec{r} = \frac{1}{3} a^3 + \frac{1}{2} a^3 - \frac{1}{3} a^3 + 0 = \frac{a^2}{2}$$

RHS:-

$$\int_S (\nabla \times \vec{F}) \cdot \vec{n} \, ds$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(xy) \right) - \hat{j} \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2) \right) + \hat{k} \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2) \right)$$

$$= 0 + \hat{k}(y)$$

$$\int_S \vec{y} (\vec{n} \cdot \vec{k}) \, ds = \int_S y \, dx \, dy$$

limits of $x \rightarrow 0$ to a

limits of $y \rightarrow 0$ to a

$$\int_0^a \int_0^a y \, dx \, dy = \int_0^a dx \int_0^a y \, dy$$

$$= \left[x \right]_0^a \times \left[\frac{y^2}{2} \right]_0^a$$

$$= [a-0] \left[\frac{a^2}{2} \right] = \frac{a^3}{2}$$

$\therefore LHS = RHS$

Hence Stoke's theorem is verified.

29-9-22

4) Verify Stoke's theorem for $\vec{F} = y\vec{i} + z\vec{k} + x\vec{k}$, and surface S of the part of the sphere.

$S: x^2 + y^2 + z^2 = 1$

Sol

Given, $\vec{F} = y\vec{i} + z\vec{k} + x\vec{k}$, $x^2 + y^2 + z^2 = 1$

By Stoke's theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \int_S \text{curl } \vec{F} \cdot \vec{n} \, ds \quad \text{--- (1)}$$

RHS:-

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \vec{i}(-1) - \vec{j}(1-0) + \vec{k}(-1)$$

$$= -\vec{i} - \vec{j} - \vec{k}$$

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{1}$$

$$\text{curl } \vec{F} \cdot \vec{n} = (-\vec{i} - \vec{j} - \vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= -x - y - z$$

$$= -(x + y + z)$$

Let R -xy plane

$$ds = \frac{dx \, dy}{|\vec{r} \cdot \vec{k}|}$$

$$= \frac{dx \, dy}{z} = \frac{dx \, dy}{\sqrt{1-x^2-y^2}}$$

limits:-

on xy plane $z=0$

$$x^2 + y^2 = 1 \Rightarrow y = \sqrt{1-x^2}$$

put $y=0, x=1$

$$\therefore x \rightarrow 0 \text{ to } 1, y \rightarrow 0 \text{ to } \sqrt{1-x^2}$$

$$\therefore \int_S \text{curl } \vec{F} \cdot \vec{n} \, ds = \int_{x=0}^1 \int_{y=0}^{\sqrt{1-x^2}} -(x+y+\sqrt{1-x^2-y^2}) \frac{dy \, dx}{\sqrt{1-x^2-y^2}}$$

Take $x = r \cos \theta$

$$y = r \sin \theta$$

$$dx \, dy = r \, dr \, d\theta$$

$$r \rightarrow 0 \text{ to } 1$$

$$\theta \rightarrow 0 \text{ to } 2\pi$$

$$= - \int_{r=0}^1 \int_{\theta=0}^{2\pi} (r \cos \theta + r \sin \theta + \sqrt{1-r^2}) \frac{r \, dr \, d\theta}{\sqrt{1-r^2}}$$

$$= - \int_{r=0}^1 \left[\int_{\theta=0}^{2\pi} r (\cos \theta + \sin \theta + \sqrt{1-r^2}) \, d\theta \right] \frac{r}{\sqrt{1-r^2}} \, dr$$

$$\begin{aligned}
 &= - \int_{r=0}^1 \left[\gamma (\sin \theta - \cos \theta) + \sqrt{1-r^2} \theta \right]_0^{2\pi} \frac{\gamma}{\sqrt{1-r^2}} dr \\
 S &= - \int_{r=0}^1 \left[\gamma (0-1) + 2\pi \sqrt{1-r^2} - (\gamma(-1)) \right] \frac{\gamma}{\sqrt{1-r^2}} dr \\
 &= - \int_{r=0}^1 \left[-\gamma + 2\pi \sqrt{1-r^2} + \gamma \right] \frac{\gamma}{\sqrt{1-r^2}} dr \\
 &= - \int_{r=0}^1 2\pi \sqrt{1-r^2} \times \frac{\gamma}{\sqrt{1-r^2}} dr \\
 &= -2\pi \left[\frac{r^2}{2} \right]_0^1 = -\pi //
 \end{aligned}$$

LHS:-

$$\begin{aligned}
 \oint_C \vec{F} \cdot d\vec{r} &= \int_C y dx = \int_0^{2\pi} \sin \theta (-\sin \theta) d\theta \\
 x &= \cos \theta, y = \sin \theta \\
 dx &= -\sin \theta d\theta \\
 \theta &\rightarrow 0 \text{ to } 2\pi \\
 &= - \int_0^{2\pi} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta \\
 &= -\frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} \\
 &= -\frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} = -\frac{1}{2} [2\pi]
 \end{aligned}$$

$$LHS = RHS = -\pi$$

$\therefore$ Stokes's theorem is verified.

5) By using Stoke's theorem prove that $\oint \vec{r} \cdot d\vec{r} = 0$ when C is a simple closed curve forming the boundary of a surface.

$$\text{Sol } \oint_C \vec{r} \cdot d\vec{r} = 0$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_S \text{curl } \vec{F} \cdot \vec{n} ds$$

$$\oint_C \vec{r} \cdot d\vec{r} = \int_S \text{curl } \vec{r} \cdot \vec{n} ds$$

$$= 0 \quad (\because \text{curl } \vec{r} = 0)$$

Gauss - Divergence Theorem:-

Let 'S' be a closed surface enclosed by volume 'V' if $\vec{F}$ is a continuously differentiable vector point function

$\int_S \vec{F} \cdot \vec{n} ds = \int_V \text{div } \vec{F} dv$ where $\vec{n}$ is outward drawn unit normal vector at any point of 'S'.

1) Compute $\int (ax^2 + by^2 + cz^2) ds$ over the surface of the sphere $x^2 + y^2 + z^2 = 1$.

$$\text{Given } \int_S (ax^2 + by^2 + cz^2) ds, \quad x^2 + y^2 + z^2 = 1.$$

$$\int_S \vec{F} \cdot \vec{n} ds = \int_V \text{div } \vec{F} dv$$

$$\text{G.T. } \vec{F} \cdot \vec{n} = ax^2 + by^2 + cz^2$$

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$\begin{aligned} \vec{F} \cdot (\vec{n}) &= (ax\vec{i} + by\vec{j} + cz\vec{k}) \cdot \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} \\ &= \frac{2ax^2 + 2by^2 + 2cz^2}{\sqrt{4x^2 + 4y^2 + 4z^2}} \\ &= \frac{2ax^2 + 2by^2 + 2cz^2}{2\sqrt{x^2 + y^2 + z^2}} \\ &= \frac{2ax^2 + 2by^2 + 2cz^2}{2(x\vec{i} + y\vec{j} + z\vec{k})} \\ &= a + b + c \end{aligned}$$

$$\int_V \text{div } \vec{F} \, dv = \int_V (a+b+c) \, dv$$

$$= (a+b+c) \int_V dv$$

$$= (a+b+c) \frac{4}{3} \pi r^3$$

$$= (a+b+c) \frac{4}{3} \pi (1)$$

$$= \frac{4}{3} \pi (a+b+c)$$

2) Applying Gauss divergence theorem, prove

that $\int_S \vec{r} \cdot \vec{n} \, ds = 3V$

$$\int_S \vec{r} \cdot \vec{n} \, ds = \int_S \vec{F} \cdot \vec{n} \, ds = \int_V \text{div } \vec{F} \, dv$$

$$\int_S \vec{r} \cdot \vec{n} \, ds = \int_V \text{div } \vec{r} \, dv$$

$$= 3 \int_V dv = 3V //$$

3) Evaluate $\int_S \vec{F} \cdot \vec{n} \, ds$, if $\vec{F} = xy\vec{i} + xz\vec{j} + 2yz\vec{k}$ over the tetrahedron bounded by $x=0, y=0, z=0$ and the plane $x+y+z=1$.

$$\vec{F} = xy\vec{i} + xz\vec{j} + 2yz\vec{k}$$

$$\int_S \vec{F} \cdot \vec{n} \, ds = \int_V \text{div } \vec{F} \, dv$$

$$\text{div } \vec{F} = y + x + 2y = 3y$$

$$\therefore x+y+z=1$$

$$z = 1-x-y; \quad z \rightarrow 0 \text{ to } 1-x-y$$

$$\text{put } z=0, \quad x+y=1$$

$$y = 1-x; \quad y \rightarrow 0 \text{ to } 1-x$$

$$\text{put } y=0, \quad x=1$$

$$\therefore x \rightarrow 0 \text{ to } 1$$

$$\therefore \int_S \vec{F} \cdot \vec{n} \, ds = \int_{x=0}^1 \left[\int_{y=0}^{1-x} \left[\int_{z=0}^{1-x-y} 3y \, dz \right] dy \right] dx$$

$$= 3 \int_{x=0}^1 \left[\int_{y=0}^{1-x} y [z]_0^{1-x-y} dy \right] dx$$

$$= 3 \int_{x=0}^1 \left[\int_{y=0}^{1-x} y(1-x-y) dy \right] dx$$

S =

$$\begin{aligned}
 &= 3 \int_{x=0}^1 \left[\int_{y=0}^{1-x} (1-x) y - y^2 dy \right] dx \\
 &= 3 \int_{x=0}^1 \left[(1-x) \frac{y^2}{2} - \frac{y^3}{3} \right]_0^{1-x} dx \\
 &= 3 \int_{x=0}^1 \left[(1-x) \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} \right] dx \\
 &= 3 \int_{x=0}^1 \left[(1-x)^3 \left(\frac{1}{6} \right) \right] dx = \frac{1}{2} \left[\frac{(1-x)^4}{-4} \right]_0^1 \\
 &= +\frac{1}{8} [0-1] = 1/8.
 \end{aligned}$$

4) If $\vec{n}$ is any unit outward drawn normal to any closed surface, show that $\int_V \text{div } \vec{n} dv = 1$

$$\int_V \text{div } \vec{F} dv = \int_S \vec{F} \cdot \vec{n} ds$$

$$\Rightarrow \int_V \text{div } \vec{n} dv = \int_S \vec{n} \cdot \vec{n} ds$$

$$= \int_S ds$$

$$= S$$

5) Verify Gauss divergence theorem for $\vec{F} = 4xy\vec{i} - y^2\vec{j} + yz\vec{k}$ where S is the surface of the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$.

$$\text{G.T } \vec{F} = 4xy\vec{i} - y^2\vec{j} + yz\vec{k}$$

$$\int_S \vec{F} \cdot \vec{n} ds = \int_V \text{div } \vec{F} dv \quad \text{--- (1)}$$

RHS:-

$$\text{div } \vec{F} = 4xy - 2y + y = 3y$$

$$\therefore \int_V \text{div } \vec{F} = 3 \int_{x=0}^1 \left[\int_{y=0}^1 y \left[\int_{z=0}^1 dz \right] dy \right] dx$$

$$= 3 \int_0^1 dx \cdot \int_0^1 y dy \cdot \int_0^1 dz$$

$$= 3 \left[x \right]_0^1 \left[\frac{y^2}{2} \right]_0^1 \left[z \right]_0^1$$

$$= 3(1-0) \left(\frac{1}{2} \right) (1-0)$$

$$= 3/2.$$

LHS:-

In BEFC, $n = \bar{i}$, $x = 1$

$$ds = \frac{dy dz}{|\bar{n}|} = dy dz$$

$$y \rightarrow 0 \text{ to } 1$$

$$z \rightarrow 0 \text{ to } 1$$

$$\therefore \vec{F} \cdot \vec{n} = 4xy = 4y$$

$$\int_{BEFC} \vec{F} \cdot \vec{n} ds = \int_{y=0}^1 \int_{z=0}^1 4y dy dz$$

$$= \int_{y=0}^1 4y [z]_0^1 dy$$

$$= \int_{y=0}^1 4y (1-0) dy = 2 \left[\frac{y^2}{1} \right]_0^1 = 2(1-0) = 2.$$

In AOGD, $\vec{n} = -\bar{i}$, $x = 0$, $ds = dy dz$

$$\vec{F} \cdot \vec{n} = -(4xy) = 0$$

$$\therefore \int_{AOGD} \vec{F} \cdot \vec{n} ds = 0$$

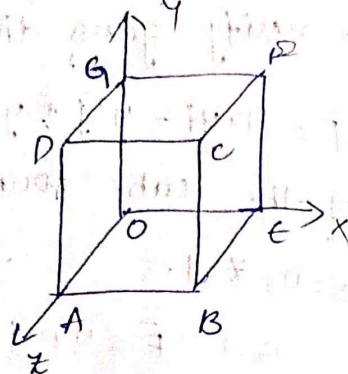
In CDFG, $y = 1$, $\vec{n} = \bar{j}$, $ds = dx dz$

$$\vec{F} \cdot \vec{n} = -y^2 = -1$$

$$\therefore \int_{CDFG} \vec{F} \cdot \vec{n} ds = \int_0^1 \int_0^1 -1 dx dz$$

$$= -1 [x]_0^1 [z]_0^1$$

$$= -1$$



On ABED, $y = 0$, $\vec{n} = -\bar{j}$, $ds = dx dz$

$$\vec{F} \cdot \vec{n} = -y^2 = 0$$

$$\int_{ABED} \vec{F} \cdot \vec{n} ds = 0$$

On ABCD, $z = 1$, $\vec{n} = \bar{k}$, $ds = dx dy$

$$\int_{ABCD} \vec{F} \cdot \vec{n} ds = \int_{x=0}^1 \int_{y=0}^1 y dx dy$$

$$= \int_{x=0}^1 \left(\frac{y^2}{2} \right)_0^1 dx = \int_{x=0}^1 \left(\frac{1}{2} - 0 \right) dx$$

$$= \frac{1}{2} (1-0) = \frac{1}{2}$$

$$= \left[\frac{x}{1} \right]_0^1 = 1$$

On EDFG, $\vec{n} = -\bar{k}$, $z = 0$

$$\vec{F} \cdot \vec{n} = 0, ds = dx dy$$

$$\therefore \int_{EDFG} \vec{F} \cdot \vec{n} ds = 0$$

$$\therefore \int_S \vec{F} \cdot \vec{n} ds = 2 + 0 - 1 + \frac{1}{2} + 0 + 0 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$LHS = RHS$$

Hence Gauss divergence theorem is verified.

6) Verify Gauss divergence theorem for

$\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ over the surface S of the solid cut off by the plane $x+y+z=a$ in the first octant.

7) Verify Gauss divergence theorem for the

vector $\vec{F} = 2x^2y\vec{i} - y^2\vec{j} + 4xz^2\vec{k}$ taken over the region of the first octant of the cylinder $y^2 + z^2 = a$ and $x=2$

Sol By Gauss divergence theorem

$$\int \vec{F} \cdot \vec{n} \, ds = \int \text{div } \vec{F} \, dv$$

Let $\phi = x + y + z - a$ is plane

$$\frac{\partial \phi}{\partial x} = 1, \frac{\partial \phi}{\partial y} = 1, \frac{\partial \phi}{\partial z} = 1$$

$$\nabla \phi = \vec{i} + \vec{j} + \vec{k}$$

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$$

→ Consider xy plane

In xy plane $z = 0$

$$\therefore x + y = a \quad y = a - x$$

$$\therefore \int_S \vec{F} \cdot \vec{n} \, ds = \iint_R \frac{\vec{F} \cdot \vec{n} \, dx \, dy}{|\vec{n} \cdot \vec{k}|} = \int_{x=0}^a \int_{y=0}^a \frac{x^2 + y^2 + z^2}{\frac{1}{\sqrt{2}}} \, dx \, dy$$

$$= \int_{x=0}^a \int_{y=0}^{a-x} x^2 + y^2 + z^2 \, dx \, dy$$

$$= \int_0^a \int_0^{a-x} (x^2 + y^2 + (a-x-y)^2) \, dx \, dy$$

$$= \int_0^a \int_0^{a-x} [x^2 + y^2 + a^2 + x^2 + y^2 - 2ax + 2xy - 2ay] \, dx \, dy$$

$$= \int_0^a \int_0^{a-x} [2x^2 + 2y^2 - 2ax + 2xy - 2ay + a^2] \, dx \, dy$$

$$= \int_0^a \left[2x^2 y + \frac{2y^3}{3} - 2axy + \frac{2xy^2}{2} - \frac{2ay^2}{2} + a^2 y \right]_0^{a-x} \, dy$$

$$= \int_0^a \left[2x^2 y + \frac{2y^3}{3} - 2axy + xy^2 - ay^2 + a^2 y \right]_0^{a-x} \, dx$$

$$= \int_0^a \left[2x^2(a-x) + \frac{2}{3}(a-x)^3 - 2ax(a-x) + x(a-x)^2 + a^2(a-x) - a(a-x)^2 \right] \, dx$$

$$= \int_0^a 2ax^2 - 2x^3 + \frac{2}{3}[a^3 - x^3 - 3a^2x + 3ax^2] - 2a^2x + 2ax^2 + x[a^2 + x^2 - 2ax] - a[a^2 + x^2 - 2ax] + a^3 - a^2x \, dx$$

$$= \int_0^a [2ax^2 - 2x^3 + \frac{2}{3}a^3 - \frac{2}{3}x^3 - 2ax^2 + 2ax^2 - 2a^2x + 2ax^2 + a^2x + x^3 - 2ax^2 - a^3 - ax^2 + 2a^3x + a^3 - a^2x] \, dx$$

$$= \int_0^a \left(-\frac{5}{3}x^3 + 3ax^2 - 2a^2x + \frac{2}{3}a^3 \right) \, dx$$

$$= \left[-\frac{5}{3} \frac{x^4}{4} + 3a \frac{x^3}{3} - 2a^2 \frac{x^2}{2} + \frac{2}{3} a^3 x \right]_0^a$$

$$= -\frac{5a^4}{12} + \frac{3a^4}{3} - \frac{2a^4}{2} + \frac{2}{3}a^3(a)$$

$$= -\frac{5a^4}{12} + \frac{2}{3}a^4 = -\frac{5a^4}{12} + \frac{8a^4}{12} = \frac{a^4}{4}$$

RHS:-

$$z = a - (x+y)$$

$$\text{div } \vec{F} = 2x + 2y + 2z$$

$$y = a - x$$

$$x = a$$

$$\int_V \text{div } \vec{F} \, dv = \int_{x=0}^a \int_{y=0}^{a-x} \left[\int_{z=0}^{a-(x+y)} (x+y+z) \, dz \right] \, dy \, dx$$

$$= 2 \int_{x=0}^a \int_{y=0}^{a-x} \left[(x+y)z + \frac{z^2}{2} \right]_0^{a-(x+y)} dy dx$$

$$= 2 \int_{x=0}^a \int_{y=0}^{a-x} \left[(x+y)(a-(x+y)) + \frac{(a-(x+y))^2}{2} \right] dy dx$$

$$= 2 \int_{x=0}^a \int_{y=0}^{a-x} \left[(ax+ay) - (x^2+y^2)^2 + \frac{(a-(x+y))^2}{2} \right] dy dx$$

$$= 2 \int_{x=0}^a \left[axy + \frac{ay^2}{2} - \frac{(x+y)^3}{3} + \frac{(a-(x+y))^3}{-6} \right]_0^{a-x} dx$$

$$= 2 \int_{x=0}^a \left[ax(a-x) + \frac{a}{2}(a-x)^2 - \frac{a^3}{3} - \frac{1}{6}(0) + \frac{x^3}{3} + \frac{1}{6}(a-x)^3 \right] dx$$

$$= 2 \int_{x=0}^a \left[a^2x - ax^2 + \frac{a}{2}(a-x)^2 - \frac{a^3}{3} + \frac{x^3}{3} + \frac{1}{6}(a-x)^3 \right] dx$$

$$= 2 \left[\frac{a^2x^2}{2} - \frac{ax^3}{3} + \frac{a}{2} \frac{(a-x)^3}{-3} - \frac{a^3}{3}x + \frac{x^4}{12} - \frac{1}{6} \frac{(a-x)^4}{-4} \right]_0^a$$

$$= 2 \left[\frac{a^4}{2} - \frac{a^4}{3} - \frac{1}{6}(0) - \frac{a^4}{3} + \frac{a^4}{12} - 0 - \left(\frac{a}{2} \times \frac{a^3}{-3} - \frac{1}{6} \frac{a^4}{4} \right) \right]$$

$$= 2 \left[\frac{a^4}{2} - \frac{a^4}{3} - \frac{a^4}{3} + \frac{a^4}{12} + \frac{a^4}{6} + \frac{a^4}{24} \right]$$

$$= 2 \left[\frac{a^4}{2} - \frac{2a^4}{3} + \frac{a^4}{12} + \frac{a^4}{6} + \frac{a^4}{24} \right]$$

$$= 2a^4 \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{12} + \frac{1}{6} + \frac{1}{24} \right]$$

$$= 2a^4 \left[\frac{12-16+2+4+1}{24} \right]$$

$$= \frac{a^4}{12} [3] = \frac{a^4}{4}$$

$\therefore LHS = RHS$

Hence Gauss divergence theorem is verified.

⑦ By Gauss divergence theorem

$$\int_S \vec{F} \cdot \vec{n} ds = \int_V \text{div } \vec{F} dv$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (2x^2y\vec{i} - y^2\vec{j} + 4xz^2\vec{k})$$

$$= 4xy - 2y + 8xz$$

$$\iiint_V \nabla \cdot \vec{F} dv = \int_{x=0}^2 \int_{y=0}^3 \int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz) dz dy dx$$

$$= \int_{x=0}^2 \int_{y=0}^3 \left[4xyz - 2yz + 8 \frac{xz^2}{2} \right]_0^{\sqrt{9-y^2}} dy dx$$

$$= \int_{x=0}^2 \int_{y=0}^3 \left[z(4xy - 2y) + 4xz^2 \right]_0^{\sqrt{9-y^2}} dy dx$$

$$= \int_{x=0}^2 \int_{y=0}^3 \left[\sqrt{9-y^2} (-2y)(1-2x) + 4x(9-y^2) \right] dy dx$$

$$= \int_0^2 \left[\frac{(1-2x)(9-y^2)^{3/2}}{3/2} \right]_0^3 + 4x \left[9y - \frac{y^3}{3} \right]_0^3 dx$$

$$= \int_0^2 \left\{ \frac{2}{3} (1-2x)(0-27) + 4x(27-9) \right\} dx$$

$$= \int_0^2 \left[-18(1-2x) + 72x \right] dx$$

$$= \left[-18 \left(x - \frac{2x^2}{2} \right) + 72 \frac{x^2}{2} \right] dx$$

$$= -18(2-4) + 36(4)$$

$$= -18(-2) + 36(4)$$

$$= 36 + 144 = 180$$

L.H.S

$$\int_S \vec{F} \cdot \vec{n} ds = \int_{S_1} \vec{F} \cdot \vec{n} ds + \int_{S_2} \vec{F} \cdot \vec{n} ds + \dots + \int_{S_5} \vec{F} \cdot \vec{n} ds$$

$$S_1 = OAB, S_2 = ECD, S_3 = OEDB, S_4 = OECA,$$

$$S_5 = ABCD$$

i) S_1 surface, on yz plane, $x=0$, $\vec{n} = -\vec{i}$

$$\vec{F} \cdot \vec{n} \Rightarrow 0(\vec{i}) - y^2\vec{j} + 4xz^2(\vec{k})$$

$$= 0$$

$$\therefore \int_{S_1} \vec{F} \cdot \vec{n} = 0$$

ii) S_2 surface: $x=2, \bar{n}=\bar{i}$

$$\bar{F} \cdot \bar{n} = 8y$$

$$\int_{S_2} \bar{F} \cdot \bar{n} ds = \int_0^3 \int_0^{\sqrt{9-z^2}} 8y dy dz = \int_0^3 8 \left[\frac{y^2}{2} \right]_0^{\sqrt{9-z^2}} dz$$

$$= \int_0^3 36 - 4z^2 dz$$

$$= 36 \left[z \right]_0^3 - 4 \left[\frac{z^3}{3} \right]_0^3$$

$$= 72$$

On S_3 $y=0, \bar{n}=-\bar{j} \therefore \bar{F} \cdot \bar{n} = 0 \therefore \int_{S_3} \bar{F} \cdot \bar{n} ds = 0$

on S_4 $z=0, \bar{n}=\bar{k}, \therefore \bar{F} \cdot \bar{n} = 0, \int_{S_4} \bar{F} \cdot \bar{n} ds = 0$

on S_5 : $y^2 + z^2 = 9, \bar{n} = \frac{\nabla(y^2 + z^2)}{|\nabla(y^2 + z^2)|}$

$$= \frac{2y\bar{i} + 2z\bar{k}}{\sqrt{4y^2 + 4z^2}}$$

$$= \frac{y\bar{i} + z\bar{k}}{2}$$

$$\bar{F} \cdot \bar{n} = \frac{-y^3 + 4xz^3}{2}$$

$$\bar{n} \cdot \bar{k} = \frac{z}{2} = \frac{1}{2} \sqrt{9-y^2}$$

$$\therefore \int_{S_5} \bar{F} \cdot \bar{n} ds = \iint_R \bar{F} \cdot \bar{n} \frac{dx dy}{|\bar{n} \cdot \bar{k}|}$$

$$= \iint_R \frac{4xz^3 - y^3}{\sqrt{9-y^2}} dx dy$$

$$= \int_{x=0}^2 \int_{y=0}^3 (4x(9-y^2) - y^3(9-y^2)^{-1/2}) dy dx$$

$$= \int_0^2 72x dx - 18 \int_0^2 dx$$

$$= 72 \left(\frac{x^2}{2} \right)_0^2 - 18(x)_0^2$$

$$= 72(2) - 18(2)$$

$$= 144 - 36$$

$$= 108$$

$$\int_{S_1} \bar{F} \cdot \bar{n} + \dots + \int_{S_5} \bar{F} \cdot \bar{n} ds = 108 + 72 = 180$$

ODEVC MID-I QUESTION BANK

SUBJECTIVE QUESTIONS

- 1) Solve $(1 + x^2) \frac{dy}{dx} + 2xy = 4x^2, y(0) = 0$
- 2) Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x \sin x$
- 3) Solve the D.E $\frac{dy}{dx} = \frac{x^2 + xy}{x^2 + y^2}$
- 4) Solve $(ydx + xdy)x \cos\left(\frac{y}{x}\right) = (xdy - ydx)y \sin\left(\frac{y}{x}\right)$
- 5) Solve the D.E $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$
- 6) If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after 36 minutes
- 7) If 30% of a radioactive substance disappears in 10 days, how long will it take for 90% of it to disappear?
- 8) The number of bacterial in a culture doubled in 2 hrs. When it will be tripled?
- 9) Solve $(D^2 + 4D + 3)y = e^{e^x}$
- 10) Solve the D.E $(D^2 - 2D + 2)y = \sin x + e^{2x}$
- 11) Solve $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$
- 12) Solve $(D^2 + 4)y = \tan 2x$ by variation of parameters.
- 13) Solve $(D^2 + 1)y = x \sin x$ by the method of variation of parameters
- 14) Solve the D.E $x^2 y'' + 3xy' + 5y = x \cos(\log x)$
- 15) Solve the D.E $(x + 2)^2 y'' - (x + 2)y' + y = 3x + 4$
- 16) Solve the D.E $(D^2 - 2D + 1)y = x^2 e^{3x} - \sin 2x + 3$
- 17) Define Laplace transform and prove that $L\{\sin at\} = \frac{a}{s^2 + a^2}$
- 18) Find $L\{e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t\}$

OBJECTIVE QUESTIONS

Chose the correct alternative

1	The D.E $(x + x^8 + ay^2)dx + (y^8 - y + bxy)dy = 0$ is exact if	[]
	a) $b = 2a$	b) $a = b$
	c) $a \neq 2b$	d) $a = 1, b = 3$
2	The order of $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} - 3y = x$ is	[]
	a) 2	b) 3
	c) 1	d) None
3	The degree of the differential equation $\left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right]^{3/2} = a \frac{d^2y}{dx^2}$ is	[]
	a) 3	b) 2
	c) 1	d) None
4	The general solution of $\frac{dy}{dx} = e^{x+y}$ is	[]
	a) $e^x + e^y = c$	b) $e^{-x} + e^y = c$
	c) $e^x + e^{-y} = c$	d) $e^{-x} - e^{-y} = c$
5	The equation $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$ is	[]
	a) Homogeneous	b) variable separable
	c) Exact	d) None
6	The general solution of $\frac{ydx - xdy}{y^2} = 0$ is	[]
	a) $xy = c$	b) $x = cy$
	c) $y = cx$	d) $x = cy^2$
7	The general solution of $\frac{xdx + ydy}{x^2 + y^2} = 0$ is	[]
	a) $\log(x + y) = c$	b) $\log(x^2 + y^2) = c$
	c) $\log(xy) = c$	d) None
8	The general solution of $(1 + x^2)dy - (1 + y^2)dx = 0$ is	[]
	a) $\tan^{-1}y - \tan^{-1}x = c$	b) $\tan^{-1}y + \tan^{-1}x = c$
	c) $\sin^{-1}y + \sin^{-1}x = c$	d) None
9	The general solution of $\frac{dy}{dx} + xy = x$ is	[]
	a) $y = 1 + ce^{-x^2/2}$	b) $y = 1 + ce^{x^2/2}$
	c) $y = x + c + e^{x^2/2}$	d) None
10	The general solution of $(1 + y^2)dx = (\tan^{-1}y - x)dy$ is	[]
	a) $x = \tan^{-1}y - 1 + ce^{-\tan^{-1}y}$	b) $x = \tan^{-1}y + 1 + ce^{-\tan^{-1}y}$
	c) $x = \tan^{-1}y$	d) None
11	The general solution of the D.E $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$ is	[]
	a) $\sin\left(\frac{y}{x}\right) = c$	b) $\sin\left(\frac{y}{x}\right) = cx$

	c) $\cos\left(\frac{y}{x}\right) = cx$	d) $\cos\left(\frac{y}{x}\right) = c$	
12	The solution of the D.E $(x + y)^2 \frac{dy}{dx} = a^2$ is given by		[]
	a) $(y + x) = \operatorname{atan}\left(\frac{y-x}{a}\right)$	b) $(y - x) = \operatorname{atan}\left(\frac{y-c}{a}\right)$	
	c) $(y - x) = \operatorname{atan}(y - c)$	d) $a(y - x) = \tan\left(y - \frac{c}{a}\right)$	
13	The integrating factor of $x \frac{dy}{dx} - y = 2x^2 \operatorname{cosec} 2x$ is		[]
	a) x	b) $\frac{1}{x}$	
	c) e^{-x}	d) None	
14	The equation $e^x dx + (xe^y + 2y)dy = 0$ is		[]
	a) Linear	b) variable's separable	
	c) Exact	d) Non-linear	
15	The orthogonal trajectories of the family of parabolas $y^2=4ax$.		[]
	a) $2x^2+y^2=k$	b) $2y^2+x^2=k$	
	c) $x^2-2y^2=k$	d) $2x^2-y^2=k$	
16	Newton's law of cooling is used to determine		[]
	a) Mass of a cooled object	b) temperature of an object after a given time	
	c) volume of a cooled object	d) pressure exerted by a cooled object	
17	Newton's Law of Cooling states that the rate at which an object cools is proportional to what?		[]
	a) Difference in temperature of the object and the temperature of its surroundings	b) Temperature of surroundings	
	c) temperature of the object	d) object's mass	
18	The differential equation $2ydx - (3y - 2x)dy = 0$		[]
	a) exact , homogeneous and linear	b) homogeneous and linear but not exact	
	c) Exact and linear but not homogeneous	d) exact and homogeneous but not linear	
19	The differential equation of the family of curve $y^2 = 4a(x + a)$ where a is arbitrary constant is _____		[]
	a) $y^2 = 4 \frac{dy}{dx} \left(x + \frac{dy}{dx}\right)$	b) $y^2 \left(\frac{dy}{dx}\right)^2 + 2xy \frac{dy}{dx} - y^2 = 0$	
	c) $y^2 = \frac{dy}{dx} + 4y$	d) $2y = \frac{dy}{dx} - 4a$	
20	The general solution of $(4D^2 + 4D + 1)y = 0$ is		[]
	a) $y = c_1 e^{-\frac{x}{2}} + c_2 e^{-\frac{x}{2}}$	b) $y = (c_1 x + c_2) e^{-\frac{x}{2}}$	
	c) $y = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{x}{2}}$	d) $y = (c_1 x + c_2) e^{x/2}$	
21	The C.F of $(D+1)(D-2)^2 y = e^{3x}$ is		[]
	a) $(c_1 + c_2 x) e^{-x} + c_3 e^{3x}$	b) $(c_1 + c_2 x) e^{2x} + c_3 e^{-x}$	
	c) $c_1 e^{-x} + c_2 e^{2x}$	d) None	

22	The P.I of $(D^2 + a^2)y = \cos ax$ is		[]
	a) $\frac{-x}{2a} \cos ax$	b) $\frac{x}{2a} \sin ax$	
	c) $x \cos ax$	d) $x \sin ax$	
23	The P.I of $(D^2 - 5D + 6)y = e^{2x}$ is		[]
	a) $-xe^{2x}$	b) xe^{2x}	
	c) e^{2x}	d) 0	
24	P.I of $(D+1)^2 y = x$ is		[]
	a) x	b) $x-2$	
	c) $(x+1)^2$	d) $(x+2)^2$	
25	$\frac{1}{D^2 + D + 1} \sin x =$		[]
	a) $\sin x$	b) $\cos x$	
	c) $\frac{1}{3} \sin x$	d) $-\cos x$	
26	The value of $\frac{1}{D-2} \sin x$ is		[]
	a) $\frac{-1}{5} (\cos x + 2 \sin x)$	b) $\frac{1}{5} (\cos x)$	
	c) $\frac{1}{5} (\sin x)$	d) $\frac{1}{5} (\cos x + 2 \sin x)$	
27	The value of $\frac{1}{D^2+4} \sin 2x$ is		[]
	a) $\frac{1}{5} \sin 2x$	b) $\frac{-1}{5} \sin^2 x$	
	c) $\frac{1}{5} \cos 2x$	d) $\frac{-x}{4} \cos 2x$	
28	$\frac{1}{D^2 - 1} e^x =$		[]
	a) $\frac{xe^x}{2}$	b) $\frac{-xe^x}{2}$	
	c) $\frac{x^2 e^x}{2}$	d) None	
29	$\frac{1}{D + 2} (x + e^x) =$		[]
	a) $\frac{-x}{2} - \frac{1}{4} + \frac{e^x}{3}$	b) $\frac{x}{2} - \frac{1}{4} + \frac{e^x}{3}$	
	c) $\frac{x}{2} - \frac{1}{4} + e^x$	d) None	
30	The C.F of $(D^2 + 4D + 5)y = 13e^x$ is		[]

	a) $e^{-2x}(c_1 \cos x + c_2 \sin x)$	b) $e^{2x}(c_1 \cos x + c_2 \sin x)$	
	c) $e^x(c_1 \cos 2x + c_2 \sin 2x)$	d) None	
31	C.F of $(D - 1)^2 y = \sin 2x$ is		[]
	a) $(c_1 + c_2 x)e^x$	b. $(c_1 + c_2 x)e^{-x}$	
	c) $c_1 x + c_2 e^x$	d) None	
32	P.I of $(D^2 - 4)y = \sin 3x$ is		[]
	a) 1/4	b) -1/13	
	c) 1/5	d) None	
33	C.F of $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = 2x \log x$ is		[]
	a) $(c_1 + c_2 x)e^x$	b) $(c_1 + c_2 \log x)x$	
	c) $(c_1 + c_2 x) \log x$	d) $(c_1 + c_2 \log x) e^x$	
34	The number of arbitrary constants in the particular solution of differential equation of second order is.....		[]
	a) 4	b) 2	
	c) 1	d) 0	
35	$\frac{1}{D} \cos x = \underline{\hspace{2cm}}$		[]
	a) $\sin x$	b) $-\sin x$	
	c) $\cos x$	d) None	
36	$L\{18\} = \underline{\hspace{2cm}} -$		[]
	a) $\frac{18}{S^2}$	b) $\frac{18}{S}$	
	c) $\frac{1}{18S}$	d) 18	
37	$L\{e^{2t}\} =$		[]
	a) $\frac{1}{s-2}$	b) $\frac{1}{s-2}$	
	c) $\frac{1}{s^2}$	d) $\frac{2}{s}$	
38	$L\{t\} =$		[]
	a) $\frac{1}{s^2}$	b) s^2	

	c) $\frac{2}{s}$	d) $\frac{2}{s^2}$	
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Fill in the Blanks

1	$f(x, y) = \frac{x^3 - y^3}{x + y}$ is a homogeneous function of degree _____
2	Integrating factor of the D.E $\frac{dx}{dy} + \frac{3x}{y} = \frac{1}{y^2}$ is _____
3	$(x^3 - 3xy^2)dx + (y^3 - 2x^2y)dy$ is an exact D.E ? true or false _____
4	Integrating factor of the D.E $\sec^2 y \frac{dy}{dx} + 2x \tan(y) = x^3$ is _____
5	the order of a differential equation whose general solution is $y = A \sin x + B \cos x$ is _____ (A, B are arbitrary constants.)
6	An integrating factor of differential equation $\frac{dx}{dy} = \frac{1}{x + y + 2}$ is _____
7	General solution of the differential equation $x \frac{dx}{dy} + y = 0$ is _____
8	P.I of $y'' - 3y' + 2y = 12$ is
9	If $f(D) = D^2 - 2$, $\frac{1}{f(D)} e^{2x}$ is
10	The general solution of $(D^2 - D - 2)x = 0$ is $x = c_1 e^t + c_2 e^{-2t}$ (True or False)
11	The C.F of $(D^2 + 1)y = 0$ is _____
12	$\frac{1}{D^2 + a^2} (\sin ax)$ is _____
13	$L\{t^2 + 3t + 10\} = \text{-----}$
14	$L\{\sqrt{t}\} =$

ODE&VC MID-II Question Bank

Subjective Questions

Q1	Find the Laplace transform of $te^{\square 3t} \cos 2t$
Q2	Find $L^{-1} \left\{ \frac{3s+7}{s^2-2s-3} \right\}$
Q3	Using convolution theorem to find $L^{-1} \left\{ \frac{s}{(s^2+4)^2} \right\}$
Q4	Using Laplace Transform, solve the differential equation $y'' - 4y' + 4y = 64 \sin 2t$, given that $y(0) = 0, y'(0) = 1$.
Q5	Using Laplace Transform, solve the differential equation $y'' + 2y' + y = 3te^{-t}$ given that $y(0) = 4, y'(0) = 0$.
Q6	Find $\text{grad} \phi$ where $\phi(x, y, z) = \log(x^2 + y^2 + z^2)$ at $(1,1,1)$.
Q7	Find the directional derivative of $\phi = xyz^2 + xz$ at $(1,1,1)$ in the direction of the normal to the surface $3xy^2 + y = z$ at $(0,1,1)$
Q8	Prove that $(y^2 - z^2 + 3yz - 2x)\bar{i} + (3xz + 2xy)\bar{j} + (3xy - 2xz + 2z)\bar{k}$ is solenoidal and irrotational
Q9	Show that $\nabla^2(r^n) = n(n+1)r^{n-2}$
Q10	Show that $\nabla \times (\bar{A} \times \bar{B}) = \bar{A}(\nabla \cdot \bar{B}) - \bar{B}(\nabla \cdot \bar{A}) + (\bar{B} \cdot \nabla)\bar{A} - (\bar{A} \cdot \nabla)\bar{B}$
Q11	Find the work done in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$.
Q12	Evaluate $\iint_S (yz\bar{i} + zx\bar{j} + xy\bar{k}) \cdot d\bar{s}$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant.
Q13	Verify Stokes theorem for $\bar{F} = (2x - y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by the projection of the xy plane.
Q14	Verify Green's theorem for $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the region defined by $y = \sqrt{x}, y = x^2$.
Q15	Use Gauss Divergence theorem to evaluate $\iint_S \bar{F} \cdot \bar{n} \, ds$ where $\bar{F} = x^3\bar{i} + y^3\bar{j} + z^3\bar{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

OBJECTIVE QUESTIONS

I. Chose the correct alternative

1	$L\{18\} = \text{-----}$		[]
	a) $\frac{18}{s^2}$	b) $\frac{18}{s}$	
	c) $\frac{1}{18s}$	d) 18	
2	$L\{t^2 + 3t + 10\} = \text{-----}$		[]
	a) $\frac{2}{s^3} - \frac{3}{s^2} + \frac{10}{s}$	b) $\frac{2}{s^3} + \frac{3}{s^2} - \frac{10}{s}$	
	c) $\frac{2}{s^3} - \frac{3}{s^2} - \frac{10}{s}$	d) $\frac{2}{s^3} + \frac{3}{s^2} + \frac{10}{s}$	
3	$L\{e^{-2t} \sin 3t\} = \text{-----}$		[]
	a) $\frac{1}{s^2 + 4s - 13}$	b) $\frac{3}{s^2 + 6s + 13}$	
	c) $\frac{3}{s^2 + 4s + 13}$	d) $\frac{3}{s^2 + 4s + 25}$	
4	When $s > a$, $L\{e^{at} t^n\} = \text{-----}$		[]
	a) $\frac{n!}{(s+a)^n}$	b) $\frac{n!}{(s-a)^n}$	
	c) $\frac{n}{(s-a)^n}$	d) $\frac{n!}{(s-a)^{n+1}}$	
5	If $L\{f(t)\} = \bar{f}(s)$, then $L\left\{f\left(\frac{t}{a}\right)\right\} = \text{-----}$		[]
	a) $\bar{f}(as)$	b) $a\bar{f}(s)$	
	c) $a\bar{f}(as)$	d) $a\bar{f}(a/s)$	
6	If $L\{f(t)\} = \bar{f}(s)$ then $L\{f'(t)\} = \text{-----}$		[]
	a) $s^2 \bar{f}(s) - sf(0) - f'(0)$	b) $s^2 \bar{f}(s) + sf(0) + f'(0)$	

	c) $s\bar{f}(s) - f(0) - f'(0)$	d) $s^2\bar{f}(s) - f(0) - sf'(0)$	
7	The value of the integral $\int_0^{\infty} e^{-2t} \cos 3t \, dt = \text{-----}$		[]
	a) $\frac{1}{13}$	b) $\frac{12}{13}$	
	c) $\frac{1}{12}$	d) $\frac{2}{13}$	
8	If $L\{f(t)\} = \bar{f}(s)$ then $L\{f(t-a)H(t-a)\} = \text{-----}$		[]
	a) $\frac{e^{as}\bar{f}(s)}{s}$	b) $e^{-as}\bar{f}(s)$	
	c) $e^{-as}\bar{f}(as)$	d) $\frac{e^{-s}\bar{f}(s)}{a}$	
9	$L\left\{\frac{\sin at}{t}\right\} = \text{-----}$		[]
	a) $\int_0^{\infty} \bar{f}(s) ds$	b) $\int_0^s \bar{f}(s) ds$	
	c) $\int_{-\infty}^{\infty} \bar{f}(s) ds$	d) $\int_s^{\infty} \bar{f}(s) ds$	
10	$L^{-1}\left\{\frac{1}{s^2+4}\right\} = \text{-----}$		[]
	a) $\sin 2t$	b) $\frac{1}{2}\sin 2t$	
	c) $\frac{1}{2}\cos 2t$	d) $\cos 2t$	
11	$L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = \text{-----}$		[]
	a) $e^{-t} + e^{-2t}$	b) $e^{-t} - e^{2t}$	
	c) $e^{-t} - e^{-2t}$	d) $e^t + e^{2t}$	

12	If n is a positive integer ,then $L^{-1}\left\{\frac{1}{s^{n+1}}\right\}$		[]
	a) $\frac{t^n}{n!}$	b) $\frac{t^n}{n}$	
	c) $\frac{t^{n+1}}{n!}$	d) $\frac{t^{n+1}}{(n+1)!}$	
13	$L^{-1}\left\{\frac{1}{s^2 + 4s + 13}\right\} =$		[]
	a) $\frac{1}{3}e^{-2t} \sin 3t$	b) $\frac{1}{3}e^{2t} \sin 3t$	
	c) $\frac{1}{2}e^{-3t} \sin 2t$	d) $\frac{1}{2}e^{-2t} \sin 2t$	
14	$L^{-1}\left\{\frac{2s}{(s^2 + 1)^2}\right\} =$		[]
	a) $\frac{t}{2} \sin t$	b) $t \sin t$	
	c) $t^2 \sin t$	d) $\frac{t^2}{2} \cos t$	
15	$L^{-1}\left\{\frac{ke^{-as}}{s^2 + k^2}\right\} =$		[]
	a) $\sin kt$	b) $\cos kt$	
	c) $u(t-a) \sin kt$	d) $u(t-a) \sin k(t-a)$	
16	$L^{-1}\left\{\frac{1}{s^n}\right\}$ is possible only when n is		[]
	a) positive integer	b) zero	
	c) negative integer	d) negative rational	
17	$L^{-1}\left\{\frac{1}{\sqrt{s+a}}\right\} =$		[]
	a) $\frac{e^{at}}{\sqrt{\pi t}}$	b) $\frac{e^{-at}}{\sqrt{\pi t}}$	

	c) $\frac{e^{-at}}{\sqrt{\pi}}$	d) $\frac{e^{-at}}{\sqrt{2\pi t}}$	
18	$\phi = x^3 + y^3 + z^3 - 3xyz$ then $\text{curl}(\text{grad } \phi) =$		[]
	a) 1	b) 0	
	c) $6x + 6y + 6z$	d) $x + y + z$	
19	If $\vec{F} = xy^2\vec{i} + 3xyz^2\vec{j} + x^2y^4z\vec{k}$ then $\text{div}(\text{curl } \vec{F})$ at $(1, -1, 2)$		[]
	a) 0	b) 28	
	c) 0	d) 31	
20	$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is		[]
	a) Solenoidal vector	b) Irrotational Vector	
	c) Both (a) & (b)	d) None	
21	If $\vec{a}$ is a constant vector then $\text{curl}(\vec{r} \times \vec{a}) =$		[]
	a) $2\vec{a}$	b) $\vec{0}$	
	c) $\vec{a}$	d) $-2\vec{a}$	
22	If $\vec{a}$ and $\vec{b}$ are irrotational vectors, then $\vec{a} \times \vec{b}$ is		[]
	a) Solenoidal vector	b) Irrotational Vector	
	c) Free vector	d) None	
23	If $\phi(x, y, z) = c$ is a surface then $\nabla\phi$ is		[]
	a) Normal to $\phi = c$	b) Tangent to $\phi = c$	
	c) binormal to $\phi = c$	d) None	
24	If $\phi = ax^2 + by^2 + cz^2$ satisfies Laplacian equation, then $a + b + c$ is		[]
	a) 1	b) 0	
	c) 2	d) None	
25	If $\vec{F} = x(y+z)\vec{i} + y(z+x)\vec{j} + z(x+y)\vec{k}$ then $\text{div } \vec{F}$ is		[]

	a) $x+y+z$	b) $3(x+y+z)$	
	c) $2(x+y+z)$	d) None	
26	If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $\nabla^2(\log r)$ is		[]
	a) 0	b) $\frac{1}{r^2}$	
	c) $x+y+z$	d) None	
27	If $u = x^2 + y^2 + z^2$ and $V = x\vec{i} + y\vec{j} + z\vec{k}$ then $\nabla \cdot (uV)$ is		[]
	a) 0	b) $2u$	
	c) $3u$	d) $5u$	
28	$\vec{f} = f_1(y,z)\vec{i} + f_2(z,x)\vec{j} + f_3(x,y)\vec{k}$ then $\vec{f}$ is		[]
	a) Irrotational vector	b) Solenoidal Vector	
	c) Both (a) & (b)	d) None	
29	$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ a constant Vector is		[]
	a) Solenoidal vector	b) Irrotational Vector	
	c) Both (a) & (b)	d) None	
30	$\nabla \cdot (\nabla f \times \nabla g) =$		[]
	a) 1	b) 0	
	c) Can't find without f & g	d) None	
31	$\nabla \times (\nabla \times \vec{a}) =$		[]
	a) $\nabla^2 \vec{a}$	b) $\nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a}$	
	c) $2\vec{a}$	d) $\vec{0}$	
32	If $\vec{f} = (z+3y)\vec{i} + (y+kz)\vec{j} + (x-2z)\vec{k}$ is solenoidal vector then k is		[]
	a) 2	b) 0	
	c) -2	d) None	

33	If S is any closed surface enclosing a volume V and $\vec{F} = 2x\vec{i} + 2y\vec{j} + 3z\vec{k}$ then $\iint_S \vec{F} \cdot \vec{n} dS =$		[]
	a) V	b) 3V	
	c) 6V	d) 7V	
34	If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $\oint_C \vec{r} \cdot d\vec{r} =$		[]
	a) 0	b) $\vec{r}$	
	c) x	d) None	
35	$\int \vec{r} \cdot \vec{n} dS =$		[]
	a) V	b) 3V	
	c) 4V	d) None	
36	The value of the line integral $\int \text{grad}(x+y-z) d\vec{r}$ from (0,1,-1) to (1,2,0)		[]
	a) -1	b) 0	
	c) 2	d) 3	
37	$\int \left(\frac{d\vec{r}}{dt} \cdot \vec{s} + \vec{r} \cdot \frac{d\vec{s}}{dt} \right) dt =$		[]
	a) $\vec{r} \times \vec{s}$	b) $\vec{r} \cdot \vec{s}$	
	c) $\frac{\vec{r}}{\vec{s}}$	d) None	
38	If $\vec{F} = x^2\vec{i} + y^2\vec{j}$, $\int_C \vec{F} \cdot d\vec{r}$, where $C: y = x^2$ in the xy – plane		[]
	a) $\frac{2}{3}$	b) $-\frac{2}{3}$	
	c) $\frac{3}{2}$	d) 0	
39	The work done in moving a particle in the force field $\vec{F} = 3x^2\vec{i} + \vec{j} + z\vec{k}$ is		[]
	a) $-\frac{27}{2}$	b) $\frac{27}{2}$	
	c) 27	d) 0	

40	A scalar potential ϕ has the following gradient $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$ then $\int_C \nabla\phi \cdot d\bar{r} = \underline{\hspace{2cm}}$ where $C: x=t, y=t^2, z=3t^2$ & $1 \leq t \leq 3$		[]
	a) 320	b) 540	
	c) 726	d) 620	
41	The line integral of the vector field $\bar{F} = 5xz\bar{i} + (3x^2 + 2y)\bar{j} + x^2z\bar{k}$ along the path from (0,0,0) to (1,1,1) parameterized by (t,t ² ,t) is		[]
	a) $\frac{51}{12}$	b) $\frac{53}{12}$	
	c) $\frac{57}{12}$	d) 0	
42	If $\bar{F} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$, then $\oint_C \bar{F} \cdot d\bar{r}$, where C: enclosed by the lines $x = \pm 1, y = \pm 1$		[]
	a) 21	b) 0	
	c) $\frac{8}{3}$	d) None	
43	If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then $\oint_C \bar{r} \cdot d\bar{r} =$		[]
	a) 0	b) $\bar{r}$	
	c) x	d) y	
44	A necessary and sufficient condition that the line integral $\oint_C \bar{A} \cdot d\bar{r} = 0$ for every closed curve C is that		[]
	a) $\text{div } A = 0$	b) $\text{div } A \neq 0$	
	c) $\text{curl } A = 0$	d) $\text{curl } A \neq 0$	
45	$\int \bar{r} \times \bar{n} \, dS =$		[]
	a) 0	b) 1	
	c) r	d) $\bar{r}$	
46	The value of the line integral $\int \nabla(x+y+2z) \cdot d\bar{r}$ from (0,0,0) to (1,2,3) is		[]

	a) 2	b) 3	
	c) 5	d) 9	
47	$\int_V \nabla \times \bar{F} dV =$		[]
	a) $\int_S \bar{n} \times \bar{F} dS$	b) V	
	c) S	d) 0	

II. Fill in the Blanks

1	$L\{\sin t \cos t\} =$ _____
2	$L\{t^3 e^{2t}\} =$ _____
3	The conditions under which the Laplace transform of f(t) exists are _____
4	$L\left\{\frac{1-e^t}{t}\right\} =$ _____
5	$L^{-1}\left\{\frac{1}{(s+a)^3}\right\} =$ _____
6	$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} =$ _____
7	$L^{-1}\left\{\frac{1}{s^2+8s+16}\right\} =$ _____
8	If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\left\{\int_s^\infty \bar{f}(s) ds\right\} =$ _____
9	The Convolution product $f(t)*g(t) =$ _____
10	$L^{-1}\left\{\frac{1}{\sqrt{s+4}}\right\} =$ _____
11	$L\{e^{3t} \cos 5t\} =$ _____
12	The value of $\int_0^\infty t e^{-3t} \sin t dt =$ _____

13	$L\{\cos(at+b)\} = \underline{\hspace{2cm}}$
14	$L^{-1}\left\{\frac{e^{-3s}}{s^3}\right\} = \underline{\hspace{2cm}}$
15	$L^{-1}\left\{\frac{1}{4s^2-25}\right\} = \underline{\hspace{2cm}}$
16	Physical interpretation of $\nabla\phi$ is $\underline{\hspace{2cm}}$
17	Directional derivative of ϕ in the direction of normal to the surface ϕ is $\underline{\hspace{2cm}}$
18	If $\bar{f} = \nabla\phi$ for some ϕ then $\bar{f}$ is $\underline{\hspace{2cm}}$
19	If $\phi = xy^2 + 3xyz - x^2$ then $\nabla \times (\nabla\phi) = \underline{\hspace{2cm}}$
20	$\text{grad}(r^n) = \underline{\hspace{2cm}}$
21	$\int_S \bar{r} \cdot \bar{n} dS = \underline{\hspace{2cm}}$
22	$\int_C 2zdx + 2ydy + 2xdz = \underline{\hspace{2cm}},$ where x,y,z are real and C be the straight line segment from point A: (0,2,1) to the point B: (4,1,-1)
23	$\iint_S \text{curl } \bar{F} \cdot \bar{n} dS =$ For any closed surface 'S',
24	$\iint xdydx + ydzdx + zdx dy = \underline{\hspace{2cm}},$ where $S : x^2 + y^2 + z^2 = a^2$
25	$\int_V \text{div } \bar{n} dV =$ If $\bar{n}$ is the unit outward drawn normal to any closed surface then

ODEVC UINTWISE PREVIOUS QUESTIONS

UNIT-I

(First order and first degree DE)

1. Find the integrating factor of $\frac{dy}{dx} + 2xy = e^{-x^2}$
2. Solve $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2, y(0) = 0$
3. If 30% of a radioactive substance disappears in 10 days, how long will it take for 90% of it to disappear?
4. Solve $(y + y^2)dx + xydy = 0$
5. Solve $(x + 2y^3)\frac{dy}{dx} = y$
6. State the Newton's law of cooling.
7. Find the integrating factor of the D.E of $\frac{dy}{dx} - y\sin 2x = \cot x$
8. Solve the D.E $(x^2 + 2\sin y)dx + (2x\cos y + y)dy = 0$
9. The temperature of the surrounding air is 20°C . The temperature of a hot body reduces from 100 to 70°C in 1 hour. Find the temperature of the body after 2 hours.
10. Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x \sin x$.
11. If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after 24 minutes.
12. Solve $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$
13. Radium decomposes at a rate proportional to the amount present. If $p\%$ of the original amount disappears in 1 year, how much will remain at the end of 21 years.
14. The temperature of the hot body reduces from 120°C to 80°C in 12 minutes. Find when the temperature will be reduced to 40°C if the temperature of surrounding air is 30°C
15. Solve the D.E $\frac{dy}{dx} - \frac{y}{x} = x^3$

16. The number of bacterial in a culture doubled in 2 hrs. When it will be tripled?
17. Solve the D.E $(2y - x^3)dx + xdy = 0$
18. State law of natural growth.
19. Find the equation of the curve passing through the point (1,1) whose D.E is $(y - yx)dx + (x + xy)dy = 0$
20. Solve the D.E $(hx + by + f)dy + (ax + hy + g)dx = 0$
21. Solve the D.E $x^3 \sec^2 y \frac{dy}{dx} + 3x^2 \tan y = \cos x$
22. Bacteria in a culture grow exponentially so that the initial number has doubled in 3 hours. How many times, the initial number will be present after 9 hours.
23. If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after 36 minutes.
24. Explain the method of solving Bernoulli equation
25. The number N of bacteria in a culture grows at a rate proportional to N. the value of N was initially 100 increased to 332 in one hour. What was the value of N after $1\frac{1}{2}$ hour.
26. Solve $(x - y)dx - dy = 0, y(0) = 2$
27. Solve the D.E $(e^y + 1)\cos x \, dx + e^y \sin x \, dy = 0$
28. Solve the D.E $\frac{dy}{dx} = \frac{x^2 + xy}{x^2 + y^2}$
29. Solve the D.E $y(xy + e^x)dx - e^x dy = 0$
30. Solve the D.E $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$
31. Solve $x \frac{dy}{dx} + y = x^3 y^6$
32. Solve the D.E $(1 + y^2) + (x - e^{\tan^{-1} y}) \frac{dy}{dx} = 0$
33. A copper ball is heated to a temperature of 80°C . then at time $t = 0$, it is placed in water which is maintained at 30°C . If at $t = 3$ minutes, the

temperature of the ball is reduced to 50°C then find time at which the

temperature of the ball is 40°C

34. Solve the D.E $x \log x \frac{dy}{dx} + y = 2 \log x$

35. Solve the D.E $x(x-1) \frac{dy}{dx} - y = x^2(x-1)^3$

36. A bacterial culture growing exponentially increases from 100 to 400 grams in 10 hours. How much was present after 3 hours

37. Solve $\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}} \left(1 - \frac{x}{y}\right) dy = 0$

38. Solve the D.E $\frac{dy}{dx} + \frac{y}{x \log x} = \frac{\sin 2x}{\log x}$

39. Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

40. Solve $(\sec x \tan x \tan y - e^x)dx + \sec x \sec^2 y dy = 0$

41. Solve $(x + y - 1)dy - (x - y - 2)dx = 0$

42. Solve $(ydx + xdy)xc \cot \left(\frac{y}{x}\right) = (xdy - ydx)ys \sin \left(\frac{y}{x}\right)$

43. A body kept in air with temperature 25°C cools from 140 to 80°C in 20 minutes. Find when the body cools down to 35°C

44. Solve the D.E $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

45. Solve the D.E $\frac{dy}{dx} + (y-1)\cos x = e^{-\sin x} \cos^2 x$

46. If $(x+y)^n$ is an integrating factor of $(4x^2 + 2xy + 6y)dx + (2x^2 + 9y + 3x)dy = 0$. find n and solve the equation

47. Solve $x^2 y dx - (x^3 + y^3)dy = 0$

48. Solve the D.E $dr + (2r \cot \theta + \sin 2\theta)d\theta = 0$

49. Solve the differential equation $\frac{dy}{dx} = \frac{\sqrt{1+y^2}}{\sqrt{1+x^2}}$

50. Solve $x \frac{dy}{dx} + y = x^2 y^3$

51. Solve $\frac{dy}{dx} + \frac{y}{x} = \sin 2x$

52. Radium decomposes at a rate proportional to the amount present. If p percent of amount disappears in one year, what percent of amount of radium will be left after 2 years?

UNIT-II
(Higher order linear DE)

1. Solve $(D^3 - 4D^2)y = 5$
2. Solve $\frac{1}{D^2} x^4$
3. Solve $(D^2 + 4)y = \tan 2x$ by variation of parameters.
4. Solve $(D^3 + 4D)y = 5 + \sin 2x$
5. Solve $(D^2 + 4D + 3)y = e^{e^x}$
6. Solve $(D^2 + 1)y = x^2 \sin 2x$
7. Solve the D.E $(D^2 + 2D + 1)y = 0$
8. Find the P.I of $(D^2 + 5D + 6)y = 1 + 2x + x^2$
9. Solve the D.E $(D^2 - 2D + 2)y = \sin x + e^{2x}$
10. Solve the D.E $(x + a)^2 y'' - 4(x + a)y' + 6y = x$
11. Solve the D.E $(D^2 - 4)y = \cosh(2x - 1) + e^{2x}$
12. Solve the D.E $x^2 y'' - xy' + 4y = x^2 \sin(\log x)$
13. Solve $\frac{d^2 y}{dx^2} - (a + b)\frac{dy}{dx} + aby = e^{ax} + e^{bx}$
14. Solve $(D^2 - 1)y = e^x \cos x$.
15. Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x$.
16. Solve $x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = 2 \log x$
17. Solve the D.E $(D^2 - 3D + 2)y = 2 \cos(2x + 3) + 2e^x$
18. Solve the D.E $(x + 2)^2 y'' - (x + 2)y' + y = 3x + 4$.
19. Solve $(D^2 + 4)y = \cos^2 x + x^4$
20. Solve the D.E $x^2 y'' + 3xy' + 5y = x \cos(\log x)$
21. Find the solution of the D.E $(D^2 + 4)y = 0$
22. Solve the D.E $(D^2 - 5D + 6)y = e^x \sin x$
23. Solve $(D^2 + 1)y = x \sin x$ by the method of variation of parameters
24. Solve $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$
25. Find the general solution of $(4D^2 + 4D + 1)y = 0$
26. Find $\frac{1}{(D^2 - 1)} e^x$
27. Solve $(D^3 + 3D + 2)y = e^{-x} + \cos x$

28. Solve $(D^3 - 7D^2 + 14D - 8)y = e^x \cos 2x$
29. Solve $(D^2 - 4D + 13)y = 0$
30. Find $\frac{1}{(D^2-1)}(x^2 + x)$
31. Solve the D.E $(D^2 + 9)y = \cos 3x + \sin 2x$
32. Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + y = \sec x$
33. Solve the D.E $(D^2 - 3D + 4)y = 0$
34. Solve the D.E $(D^2 - 2D + 1)y = x^2 e^{3x} - \sin 2x + 3$
35. Solve $\frac{d^3x}{dt^3} - x = 0$
36. Solve $\frac{d^2y}{dx^2} + a^2y = \tan ax$
37. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \tan x$ by the method of variation of parameters
38. Find P.I of $(D^6 - D^4)y = x^2$
39. Solve by the method of variation of parameters $(D^2 - 2D)y = e^x \sin x$
40. Find P.I of $(D^2 - D - 2)y = e^{2x}$
41. Solve the D.E $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$
42. Solve $(D^2 - 1)y = x e^x \sin x$
43. Solve $(D^3 + D^2 - D - 1)y = \sin 2x$
44. Solve $(D^3 + a^2D)y = \sin ax$
45. Solve $(D^2 - 4D + 4)y = x^2 \sin x + e^{2x} + 3$
46. Solve the D.E $(D^2 - 4)y = 2\cos^2 x$
47. Solve the D.E $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \sin x$
48. Solve $(x^2 D^2 - 3xD + 1)y = \frac{\log x \sin(\log x + 1)}{x}$
49. Solve the D.E $(D^2 + 1)y = \sin x \sin 2x$
50. Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$
51. Solve $(2x - 1)^3 \frac{d^3y}{dx^3} + (2x - 1) \frac{dy}{dx} - 2y = x$
52. Solve $(D^2 + 5D + 6)y = e^{-2x} \sec^2 x (1 + 2\tan x)$
53. Solve $y'' + 4y' + 20y = 23\sin x - 15\cos x$, where $y(0) = y'(0) = -1$

54. Solve $(x^2 D^2 + 4xD + 2)y = x^2 + \frac{1}{x^2}$

55. Solve $(xD^2 + D - \frac{1}{x})y = -a x^2, x \neq 0$

UNIT-III

(Laplace and inverse Laplace transforms)

SHORT ANSWER QUESTIONS:

1. Define Laplace transform
2. Define exponential order
3. Find $L\{t^n\}$ where n is a +ve integer
4. Find $L\{\sin at\}$
5. Find $L\{\cos^3 2t\}$
6. Find $L\{e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t\}$
7. Find the L.T of $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$
8. State first shifting theorem
9. If $L\{f(t)\} = \frac{9s^2 - 12s + 15}{(s-1)^3}$ find $L\{f(3t)\}$ using change of scale property
10. Prove $L\{f^{(n)}(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$
11. Evaluate $L\{t \sin 3t \cos 2t\}$
12. Find $L\left\{\frac{\sin 3t \cos t}{t}\right\}$
13. Define Heaviside's unit function
14. Find $L^{-1}\left\{\frac{3s-8}{4s^2+25}\right\}$
15. Find $L^{-1}\left\{\frac{3(s^2-2)^2}{2s^2}\right\}$
16. Find $L^{-1}\left\{\frac{4}{(s+1)(s+2)}\right\}$
17. Find $L^{-1}\left\{\frac{2s+3}{s^2+2s+2}\right\}$
18. Evaluate $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$
19. Find $L^{-1}\left\{\frac{e^{-2s}}{s^2+4s+5}\right\}$
20. Evaluate $L^{-1}\{\cot^{-1} s\}$
21. Find $L^{-1}\left\{\frac{s}{(s-4)^5}\right\}$
22. State Convolution theorem
23. Find $L^{-1}\left\{\frac{2s^2-4s+5}{s^3}\right\}$
24. Find $\int_0^\infty t e^{-3t} \sin t dt$

25. Find $L \left\{ \frac{\sin t}{t} \right\}$
 26. Define an inverse Laplace transform of a function
 27. Find $L^{-1} \left\{ \log \left(\frac{s+3}{s+2} \right) \right\}$
 28. Find $L^{-1} \left\{ \frac{1}{(s-a)^2 + b^2} \right\}$

LONG ANSWER QUESTIONS:

- State and prove first shifting theorem
- Find Laplace transform of (i) $e^{-3t}(2\cos 5t - 3\sin 5t)$ (ii) $e^{3t} \sin^2 t$
- State and prove second shifting theorem
- Find the Laplace transform of $g(t)$, where $g(t) = f(x) = \begin{cases} \cos \left(t - \frac{2\pi}{3} \right), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$
- If $L \{f(t)\} = \bar{f}(s)$, then $L \{f(at)\} = \frac{1}{a} \bar{f}(s/a)$
- If $L \{f(t)\} = \bar{f}(s)$, then $L \left\{ \int_0^t f(t) dt \right\} = \frac{1}{s} \bar{f}(s)$
- If $L \{f(t)\} = \bar{f}(s)$, then $L \{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} (\bar{f}(s))$ where $n=1, 2, 3, \dots$
- If $L \{f(t)\} = \bar{f}(s)$, then $L \left\{ \frac{f(t)}{t} \right\} = \int_s^\infty \bar{f}(s) ds$
- Find $L \left\{ \frac{\cos 4t \sin 2t}{t} \right\}$
- Using L.T, evaluate $\int_0^\infty \frac{e^{-t} - e^{-2t}}{t} dt$
- Using L.T, evaluate $\int_0^\infty t^2 e^{-4t} \sin 2t dt$
- Find $L \{f(t)\}$ where $f(t)$ is given by $f(t) = \begin{cases} t & , 0 < t < b \\ 2b - t & , b < t < 2b \end{cases}$,
 $2b$ being the period of $f(t)$
- Find $L^{-1} \left\{ \frac{s}{(s^2+1)(s^2+9)(s^2+25)} \right\}$
- Find $L^{-1} \left\{ \frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6} \right\}$
- Find $L^{-1} \left\{ \frac{s+3}{s^2 - 10s + 29} \right\}$
- Find $L^{-1} \left\{ \frac{1}{s(s^2-1)(s^2+1)} \right\}$
- Find $L^{-1} \left\{ \log \left(\frac{s+1}{s-1} \right) \right\}$
- Find $L^{-1} \left\{ \frac{s+3}{(s^2+6s+13)^2} \right\}$
- Using the Convolution theorem find $L^{-1} \left\{ \frac{s}{(s^2+a^2)^2} \right\}$

20. Using the Convolution theorem find $L^{-1} \left\{ \frac{1}{s(s^2+2s+2)} \right\}$
21. Using Laplace transform, solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t} \sin t$, given that $y(0)=0$, $y'(0)=1$
22. Using Laplace transform, solve $(D^3 - D^2 + 4D - 4)y = 68 = e^x \sin 2x$, $y=1, Dy=-19, D^2y = -37$ at $x=0$
23. Using Laplace transform, solve $y(t)=1 - e^{-t} + \int_0^t y(t-u) \sin u \, du$
24. Solve $\frac{dy}{dt} + 3y + 2 \int_0^t y(t) \, dt = t$ by Laplace transform method
25. Solve $(D^2+9)y = \cos 2t$, $y(0) = 1$, $y(\frac{\pi}{2}) = -1$

UNIT-IV

(Vector differentiation)

SHORT ANSWER QUESTIONS:

1. If ϕ satisfies laplacian equation, show that $\nabla \phi$ is solenoidal
2. Find curl of $x^2i + yj + zk$
3. If $\vec{a}=a_1i + a_2j + a_3k$ and $\vec{r}=xi + yj + zk$ find $\nabla(\vec{r} \cdot \vec{a})$
4. Find n , if $\vec{f} = r^n \vec{r}$ is solenoidal
5. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2,1,2)$
6. Prove that $\vec{f} = yz\vec{i} - zx\vec{j} + yx\vec{k}$ is irrotational
7. What is solenoidal vector P.T($\nabla f \times \nabla g$) is solenoidal for scalar functions f and g
8. Find $\nabla(x^2 + y^2z)$
9. If $\vec{f} = x\vec{i} - y^2\vec{j} + z^3\vec{k}$, find curl $\vec{f}$
10. Find grad ϕ when ϕ is given by $\phi = 3xyz^3 - yz$
11. Find curl($x\vec{i} + y\vec{j} + z\vec{k}$)
12. Find the directional derivative of $xyz^2 + xz$ at $(1,1,1)$ in a direction of the normal to the surface $3xy^2 + y = z$ at $(0,1,1)$
13. Find a unit normal vector to the surface $x^2 + y^2 + 2z^2 = 26$ at the point $(2,2,3)$
14. Find a vector normal to the surface $xyz^2 = 20$ at the point $(1,1,2)$
15. Find the magnitude of the gradient of the function $f(x,y,z) = xyz^3$ at $(1,0,2)$
16. Find the curl of the gradient of scalar field $V = 2x^2y + 3y^2z + 4z^2x$
17. Find the divergence of the vector field $\vec{A}$ at $(1,-1,1)$

Where $\vec{A} = x^2 z \vec{i} + xy \vec{j} - z^2 y \vec{k}$

LONG ANSWER QUESTIONS:

1. Prove that the function $\vec{f} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ is irrotational and find scalar potential function corresponding to it
2. With usual notations of vector calculus ,prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$
3. Find the directional derivative of $\phi = x^2 yz + 4xz^2$ at $(1, -2, -1)$ in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$
4. If $\nabla\phi = 2xyz\vec{i} + x^2 z\vec{j} + x^2 y\vec{k}$ find the scalar potential
5. Evaluate $\nabla^2 f(r)$ where $\vec{r} = xi + yj + zk$
6. Show that $\text{div}(r^n \vec{r}) = (n + 3)r^{-n}$
7. If $\vec{f} = \frac{\vec{r}}{r}$, find $\text{grad}(\text{div} \vec{f})$
8. Show that $\text{div}(\vec{a} \times \vec{b}) = \vec{b} \cdot \text{curl} \vec{a} - \vec{a} \cdot \text{curl} \vec{b}$
9. Find the gradient of the scalar function $f(x,y,z) = x^2 y^2 + xy^2 - z^2$ at $(3,1,1)$
10. Show that $\nabla^2(r^n) = n(n + 1)r^{n-2}$
11. If $f = (x^2 + y^2 + z^2)^{-n}$ find $\text{div}(\text{grad } f)$ and determine n if $\text{div}(\text{grad } f) = 0$
12. Show that the vector $\vec{f} = (x + 3y)\vec{i} + (y - 3z)\vec{j} + (x - 2y)\vec{k}$ is solenoidal and also find $\vec{f} \cdot \text{curl } \vec{f}$
13. In what direction from $(3,1,-2)$ is the directional derivative of $\phi = x^3 y^2 + yz$ maximum? Find also the magnitude of the maximum
14. Find the directional derivative of $xyz^2 + xz$ at $(1,1,1)$ in a direction of the normal to the surface $3xy^2 + y = z$ at $(0,1,1)$
15. Prove that $\text{curl}(\vec{a} \times \vec{b}) = \vec{a} \text{div } \vec{b} - \vec{b} \text{div } \vec{a} + (\vec{b} \cdot \nabla)\vec{a} - (\vec{a} \cdot \nabla)\vec{b}$
16. Prove that if $\vec{r}$ is the position vector of any point in space then $r^n \vec{r}$ is irrotational and is solenoidal if $n=3$

UNIT-V

(Vector Integration)

SHORT ANSWER QUESTIONS:

1. If $\vec{f} = (x^2 + xy^2)\vec{i} + (y^2 - x^2 y)\vec{j}$ is conservative , find the scalar potential ϕ
2. State Green's theorem, write the line integral which gives the area of a plane region
3. State Stokes's theorem

4. Find the work done by the force $3x^2 \bar{i} + (2xz - y) \bar{j} + z \bar{k}$ along the straight line joining the points (0,0,1) and (2,1,3)
5. Find the circulation of $\bar{f}$ round the curve c where $\bar{f} = (e^x \sin y) \bar{i} + (e^x \cos y) \bar{j}$ and c is the rectangle whose vertices are $(0,0), (1,0), (1, \frac{\pi}{2}), (0, \frac{\pi}{2})$
6. State Green's theorem
7. Find the work done by a force $y \bar{i} + x \bar{j}$ which displays a particle from origin to a point $(\bar{i} + \bar{j})$ along the line $y=x$

LONG ANSWER QUESTIONS:

1. Evaluate $\iint \bar{f} \cdot \bar{n} \, ds$ where $\bar{f} = (x+y)^2 \bar{i} - 2xy \bar{j} + 2yz \bar{k}$ and s is the surface of the plane $2x+y+2z=6$ in the first octant
2. Evaluate $\int \bar{f} \cdot d\bar{r}$ where $\bar{f} = x^2 \bar{i} + y \bar{j}$ and c is the curve $y = x^2$ in xy-plane from (0,0) to (1,1)
3. Verify Divergence theorem for the function $y \bar{i} + x \bar{j} + z^2 \bar{k}$ taken over the cylindrical region bounded by $x^2 + y^2 = a, z = 0, z = 2$
4. Evaluate $\int (x^2 + xy) dx + (x^2 + y^2) dy$ where c is a boundary of the region bounded by lines $x=0, x=1, y=0, y=1$
5. If $\bar{f} = (x^2 - y) \bar{i} - (2xz - y) \bar{j} + z^2 \bar{k}$ evaluate $\int \bar{f} \cdot d\bar{r}$ where c is a straight line joining the points (0,0,0) to (1,2,4)
6. Verify Green's theorem for $(x^2 + y^2) \bar{i} - 2xy \bar{j}$ taken around the rectangle bounded by the lines $x=a, x=-a, y=0, y=b$
7. Verify Gauss divergence theorem for the vector point function $\bar{f} = (x^3 - yz) \bar{i} - 2yx^2 \bar{j} + z \bar{k}$ over the cube bounded by $x=0, y=0, z=0$ and $x=a, y=a, z=a$
8. Apply Green's theorem to evaluate $\int (y^2 + xy) dx + x^2 dy$ where c is the boundary of the area enclosed by the x-axis and the upper half of the circle $x^2 + y^2 = a^2$
9. Find the work done by $\bar{f} = (2x - y - z) \bar{i} - (x + y - z) \bar{j} + (3x - 2y - 5z) \bar{k}$ along a curve c is the xy-plane given by $x^2 + y^2 = 9, z = 0$
10. Use Gauss divergence theorem, to evaluate $\iiint (x dy dz + y dz dx + z dx dy)$ where s is the portion of the plane $x+2y+3z=6$, which lies in the first octant
11. Using Green's theorem, find the area of the region in the first quadrant by the curves $y = x, y = \frac{1}{x}, y = \frac{x}{4}$
12. Evaluate $\iiint \text{div } \bar{f} \, dv$ where $\bar{f} = y \bar{i} + x \bar{j} + z^2 \bar{k}$ over the surface of the cylinder

$$x^2 + y^2 = a^2, z=0, z=h$$

13. Verify Divergence theorem for the function $2x^2y\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$ taken over the region of first octant of the cylinder $y^2 + z^2 = 9$ and $x = 2$
14. If $\bar{f} = 3x^2yz^2\bar{i} - x^3z^2\bar{j} + 2x^3yz\bar{k}$. show that $\int \bar{f} \cdot d\bar{r}$ is independent of the path of integration .hence evaluate the integral when C is any path joining (0,0,0) to (1,2,3)
15. Verify Gauss divergence theorem for $\bar{f} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ over the rectangular parallelepiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$
16. Evaluate $\oint x^2 dx + 2y dy - dz$ by Stoke's theorem where C is the curve $x^2 + y^2 = 4, z=2$