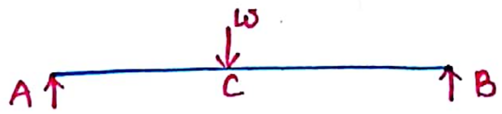
UNIT - IV

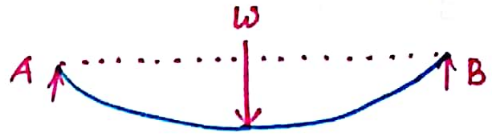
DEFLECTION OF BEAMS

* If a beam carries a uniformly distributed load or a point load, the beam is deflected from its original position.

* Due to loads acting, it will subject to Bending moment



* The radius of curvature of the deflected beam is $\frac{M}{I} = \frac{E}{R}$



where

$$R = \frac{(I \times E)}{M}$$

* The term $(I \times E)/M$ is constant, if the beam is subjected to a constant bending moment 'M'.

* This means that a beam for which, when loaded, the value of $(E \times I)/M$ is constant, will bend in a circular arc.

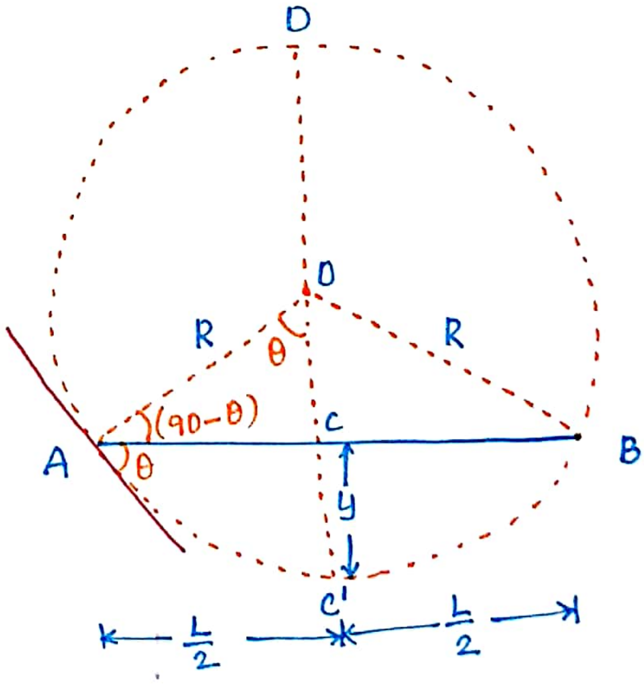
1. SLOPE & DEFLECTION OF A BEAM SUBJECTED TO UNIFORM BENDING MOMENT:

A beam AB of length 'L' is subjected to a uniform Bending moment 'M'

As the beam is subjected to a constant B.M, hence it will bend into a circular arc.

The initial position of the beam is shown by ACB,

whereas the deflected position is shown by AC'B.



Let

R – Radius of Curvature of the deflected beam.

y – Deflection of beam at the Centre.

I – Moment of inertia of the beam Section.

E – Youngs modulus for the beam material.

θ – Slope of the beam at the end 'A'.

for a practical beam the deflection 'y' is a small quantity.

Hence

$\tan\theta = \theta$, where θ is in radians.

Hence

'θ' becomes the slope as slope is

$$\frac{dy}{dx} = \tan\theta = \theta.$$

Now

$$AC = BC = \frac{L}{2}$$

Also from geometry of a circle, we know that

$$AC \times CB = DC \times CC'$$

$[\because DC = DC' - CC' = 2R - y]$.

$$\frac{L}{2} \times \frac{L}{2} = (2R - y) \times y$$

$$\frac{L^2}{4} = 2Ry - y^2$$

$\because$ for a practical beam, the deflection 'y' is so small, let us take $y^2 = 0$

$$\frac{L^2}{4} = 2Ry \Rightarrow y = \frac{L^2}{8R}$$

But from Bending moment equation, we know

$$\frac{M}{I} = \frac{E}{R} \Rightarrow R = \frac{E \times I}{M}$$

Substituting 'R' in 'y'.

$$y = \frac{L^2}{8\frac{EI}{M}}$$

$$\boxed{y = \frac{ML^2}{8EI}}$$

The equation gives the central deflection of the beam which bends into a circular arc.

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Value of slope 'θ'

from the Δ^{le} AOB, we know that

$$\sin\theta = \frac{AC}{AD} = \frac{\left(\frac{L}{2}\right)}{R} = \frac{L}{2R}$$

Since the angle 'θ' is very small, hence $\sin\theta = \theta$

$$\theta = \frac{L}{2R} \Rightarrow \theta = \frac{L}{2\times\frac{EI}{M}}$$

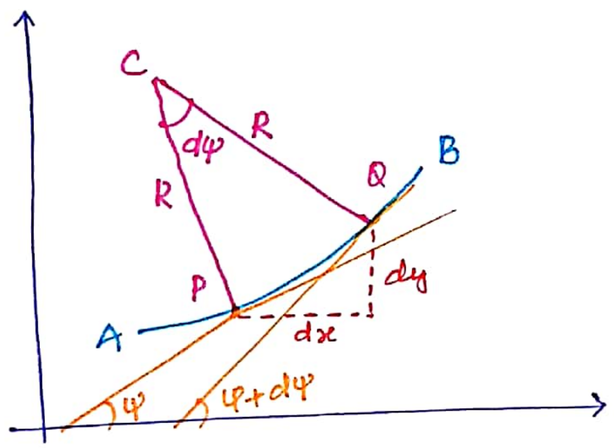
$$* \boxed{\theta = \frac{M\times L}{2EI}.} *$$

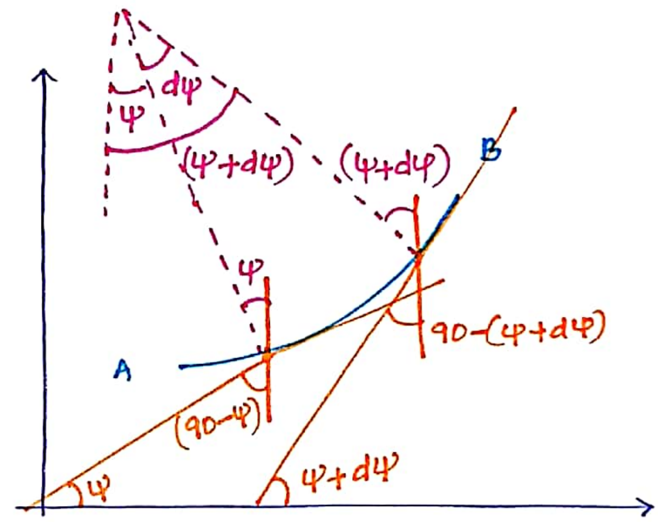
Equation gives the slope of the deflected beam at A & B.

2. RELATION B/W SLOPE, DEFLECTION & RADIUS OF CURVATURE:

Let the curve AB represents the deflection of a beam

Consider a small portion PQ of the beam, let the tangents P and Q make angle ψ and $\psi+d\psi$ with x-axis.





Normal at P and Q will meet at 'C', where 'C' is centre of curvature of 'PQ'.

Let length of PQ = 'ds'.

$$\angle PCQ = d\psi$$

$$PQ = ds = R\cdot d\psi \Rightarrow R = \frac{ds}{d\psi}$$

$$\tan\psi = \frac{dy}{dx}$$

$$\sin\psi = \frac{dy}{ds}$$

$$\cos\psi = \frac{dx}{ds}.$$

As we know

$$R = \frac{ds}{d\psi} \Rightarrow \frac{\left(\frac{ds}{dx}\right)}{\left(\frac{d\psi}{dx}\right)} = \frac{\frac{1}{\cos\psi}}{\left(\frac{d\psi}{dx}\right)}$$

$$\Rightarrow \boxed{R = \frac{\sec\psi}{\left(\frac{d\psi}{dx}\right)}}$$

Differentiate 'tan ψ' with respect to 'x'

$$\tan\psi = \frac{dy}{dx}$$

$$\sec^2\psi \cdot \frac{d\psi}{dx} = \frac{d^2y}{dx^2}$$

$$\frac{d\psi}{dx} = \frac{(d^2y/dx^2)}{\sec^2\psi}$$

Substituting $\frac{d\psi}{dx}$ in 'R'

$$R = \frac{\sec\psi}{\frac{(d^2y/dx^2)}{\sec^2\psi}}$$

$$R = \frac{\sec^3\psi}{(d^2y/dx^2)}$$

Taking Reciprocal on both sides.

$$\frac{1}{R} = \frac{(d^2y/dx^2)}{\sec^3\psi} = \frac{(d^2y/dx^2)}{(\sec^2\psi)^{3/2}}$$

$$= \frac{d^2y/dx^2}{(1+\tan^2\psi)^{3/2}}$$ [using trigonometric identity].

for a practical beam, slope 'tan ψ' is a small quantity,

Hence $\tan^2\psi$ can be neglected.

$$\boxed{\frac{1}{R} = \frac{d^2y}{dx^2}.}$$

from the Bending equation, we have

$$\frac{M}{I} = \frac{E}{R}$$

$$\boxed{\frac{1}{R} = \frac{M}{EI}}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2} \Rightarrow M = EI\frac{d^2y}{dx^2}$$

Differentiate w.r.to 'x'

$$\frac{dM}{dx} = EI.\frac{d^3y}{dx^3} = \text{Shear force}$$

$$\boxed{F = EI.\frac{d^3y}{dx^3}}$$

Differentiate w.r.to 'x'

$$\frac{dF}{dx} = EI.\frac{d^4y}{dx^4} = \text{rate of loading}$$

$$\boxed{w = EI.\frac{d^4y}{dx^4}}$$

Hence, the relation between curvature, slope, deflection, etc at any section given by.

* Deflection $= y$.
* Slope $= \frac{dy}{dx}$
* Bending moment $= EI\frac{d^2y}{dx^2}$
* Shear force $= EI\frac{d^3y}{dx^3}$
* rate of loading $= EI\frac{d^4y}{dx^4}$.

3. Methods of Determining Slope and Deflection:

i) Double integration method

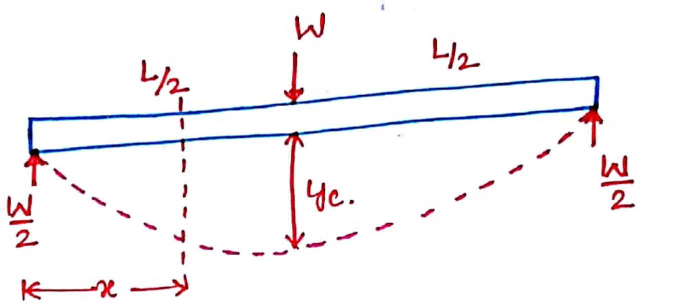
ii) Moment area method.

iii) Macaulay's method.

First two methods are used for a single load whereas the third one is used for several loads.

4. Deflection of a Simply Supported Beam with a Point Load at the Centre:

Consider a S.S. beam AB of length 'L' and carrying a point load 'W' at the centre.



Reactions at Supports are R_A, R_B ; $R_A = R_B = \frac{W}{2}$

Now

Consider a section 'X' at a distance 'x' from 'A'.

$$B.M @ x \Rightarrow M_x = R_A \cdot x$$

$$= \frac{W}{2} \times x. \quad — i)$$

But Bending moment at any section is given by

$$M = EI \frac{d^2y}{dx^2} \quad — ii)$$

Equating i) & ii) $\quad EI \frac{d^2y}{dx^2} = \frac{Wx}{2}$

On integrating we get

$$\text{Slope} = \boxed{EI \frac{dy}{dx} = \frac{W}{2} \times \frac{x^2}{2} + C_1}$$

where C_1 is the constant of integration. its value is obtained from boundary conditions.

$$@ x = \frac{L}{2} ; \quad \frac{dy}{dx} = 0$$

$$EI(0) = \frac{W}{2} \times \frac{x^2}{2} + C_1$$

$$\Rightarrow C_1 = -\frac{W}{2 \times 2} \left[\frac{L}{2}\right]^2$$

$$\Rightarrow \boxed{C_1 = -\frac{WL^2}{16}}$$

Substituting the value of C_1 in equation, we get

$$\boxed{EI\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{WL^2}{16}}$$

It is the slope equation, slope is maximum at 'A'. At A, $x=0$ and hence slope at A is

$$EI\left(\frac{dy}{dx}\right)_{at\ A} = \frac{W}{4}(0) - \frac{WL^2}{16}$$

$$EI\cdot\theta_A = -\frac{WL^2}{16}$$

$$\boxed{\theta_A = -\frac{WL^2}{16EI}}$$

Deflection at a point is obtained by integrating the slope

$$\boxed{EI \times y = \frac{W}{4}\cdot\frac{x^3}{3} - \frac{WL^2}{16}x + C_2}$$

C_2 is another constant of integration, At A, $x=0$; deflection $=0$

$$EI\times 0 = \frac{W}{4}\cdot\left(\frac{0}{3}\right) - \frac{WL^2}{16}(0) + C_2$$

$$\boxed{C_2 = 0.}$$

$$\boxed{EI\times y = \frac{Wx^3}{12} - \frac{WL^2}{16}x}$$

The maximum deflection at centre, $x = \frac{L}{2}$

$$EI\times y_c = \frac{W}{12}\left(\frac{L}{2}\right)^3 - \frac{WL^2}{16}\left(\frac{L}{2}\right)$$

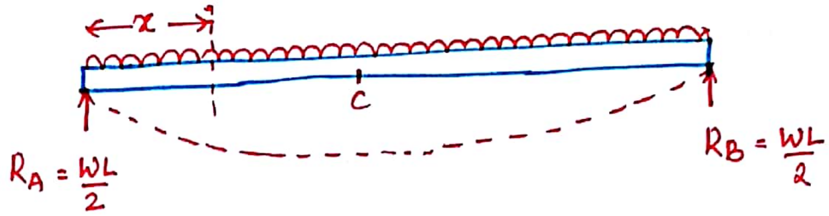
$$= \frac{WL^3}{12\times 8} - \frac{WL^3}{16\times 2}$$

$$= \frac{WL^3 - 3WL^3}{96} = -\frac{2WL^3}{96}$$

$$\boxed{y_c = -\frac{WL^3}{48EI}}$$

(Negative sign shows deflection is downwards).

5. DEFLECTION OF A SIMPLY SUPPORTED BEAM WITH A UNIFORMLY DISTRIBUTED LOAD:



$$R_A = R_B = \frac{wL}{2}$$

Consider a section 'X' at a distance 'x' from 'A'

BM @ x ; $M_x = R_A \times x - w \times x \times \frac{x}{2}$

$$M_x = \frac{wL}{2}x - \frac{w \cdot x^2}{2} \quad \text{— i)}$$

B.M @ any section is given by.

$$M = EI \frac{d^2y}{dx^2} \quad \text{— ii)}$$

equating i) & ii)

$$EI \cdot \frac{d^2y}{dx^2} = \frac{wL}{2}x - \frac{w \cdot x^2}{2}$$

Integrating the above, we get

$$\boxed{EI \cdot \frac{dy}{dx} = \frac{wL}{2} \cdot \frac{x^2}{2} - \frac{w}{2} \cdot \frac{x^3}{3} + C_1}$$

again by integrating

$$\boxed{EI \cdot y = \frac{wL}{4} \frac{x^3}{3} - \frac{w}{6} \cdot \frac{x^4}{4} + C_1 x + C_2}$$

The integral Constants C_1 & C_2 are obtained by boundary conditions

i) at $x = 0$; $y = 0$. ii) at $x = L$; $y = 0$

$$EI(0) = \frac{wL}{4}(0) - \frac{w}{6}(0) + C_1(0) + C_2$$

$$\Rightarrow \boxed{C_2 = 0}$$

$$EI(0) = \frac{wL}{12}(L^3) - \frac{w}{24}(L^4) + C_1 L + 0$$

$$0 = \frac{wL^4}{12} - \frac{wL^4}{24} + C_1 L$$

$$C_1 = -\frac{wL^3}{12} + \frac{wL^3}{24}$$

$$\boxed{C_1 = -\frac{wL^3}{24}}$$

Substituting C_1 & C_2

$$\boxed{EI\frac{dy}{dx} = \frac{w \cdot L}{4}x^2 - \frac{w \cdot}{6}x^3 - \frac{wL^3}{24}.}$$

$$\boxed{EI\,y = \frac{wL}{12}x^3 - \frac{w}{24}x^4 - \frac{wL^3}{24}x.}$$

Slope at supports

Let θ_A is slope at 'A', $x = 0$; $\frac{dy}{dx} = \theta_A$

$$EI \times \theta_A = \frac{w \cdot L}{4}(0) - \frac{w}{6}(0) - \frac{wL^3}{24}$$

$$\boxed{\theta_A = -\frac{wL^3}{24\,EI}}$$

By Symmetry

$$\boxed{\theta_B = -\frac{wL^3}{24\,EI}}$$

(or) $$\boxed{\theta_A \; \& \; \theta_B = -\frac{WL^2}{24\,EI}}$$

maximum deflection

maximum deflection occurs at centre of beam, $x = \frac{L}{2}$

$$EI \cdot y_c = \frac{wL}{12}\left[\frac{L}{2}\right]^3 - \frac{w}{24}\left[\frac{L}{2}\right]^4 - \frac{wL^3}{24}\left[\frac{L}{2}\right]$$

$$= \frac{wL^4}{96} - \frac{wL^4}{384} - \frac{wL^4}{48}$$

$$= -\frac{5wL^4}{384}$$

$$* \boxed{y_c = -\frac{5}{384}\frac{wL^4}{EI}.} *$$ (or) $$\boxed{y_c = -\frac{5}{384}\frac{WL^3}{EI}.}$$

Negative sign indicates deflection is downwards.

Q: A beam of uniform rectangular section 200 mm wide and 300 mm deep is S.S at ends. It carries a U.D.L of 9 KN/m run over the entire span of 5m. If the value of E for the beam material is 1×10^4 N/mm², find.

i) The slope at the supports
ii) Maximum deflection.

Q: A beam of length 5m and of U.D.L of 9 KN/m over entire length having a uniform rectangular section supported at its ends. Calculate the width and depth of the beam if permissible bending stress is 7 N/mm² and central deflection is not to exceed 1cm. take $E = 1 \times 10^4$ N/mm²

Sol: Given

Length, $L = 5m = 5000$ mm

U.D.L, $w = 9$ KN/m.

Total load, $W = w.L = 9 \times 5 = 45$ KN $= 45000$ N.

Bending stress, $f = 7$ N/mm².

Central deflection, $y_c = 1cm = 10mm$.

$E = 1 \times 10^4$ N/mm².

Let the dimensions be $b \times d$.

$$MOI = \frac{bd^3}{12}$$

We know $$y_c = \frac{5}{384} \frac{W.L^3}{EI}$$

$$10 = \frac{5}{384} \times \frac{45000 \times 5000^3}{1 \times 10^4 \times \frac{bd^3}{12}}$$

$$bd^3 = 878.906 \times 10^7 \text{ mm}^4.$$

$$\text{max. B.M} = \frac{wL^2}{8} = \frac{WL}{8} = \frac{45000 \times 5}{8} \times 1000 = 28125000 \text{ Nmm}$$

Now $$\frac{M}{I} = \frac{f}{y} \Rightarrow \frac{28125000}{\frac{bd^3}{12}} = \frac{7}{(d/2)} \Rightarrow bd^2 = 24107142.85 \text{ mm}^3.$$

$$\Rightarrow d = \frac{878.906 \times 10^7}{24107142.85}$$

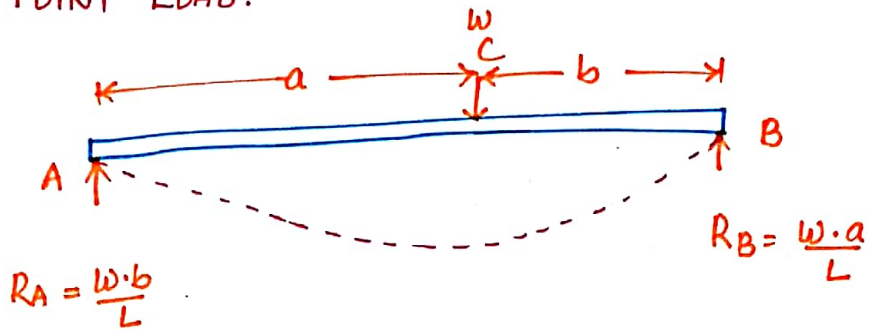
$$d = 364.58 \text{ mm}.$$

$$b \times (364.58)^2 = 24107142.85$$

$$b = 181.36 \text{ mm},$$

6. MACAULAY'S METHOD :-

DEFLECTION OF A SIMPLY SUPPORTED BEAM WITH AN ECCENTRIC POINT LOAD:



Bending moment at any section b/w A & C. at a distance 'x' from 'A'

$$M_x = R_A \times x$$

$$= \frac{W \cdot b}{L} \times x .$$

Bending moment at any section b/w C & B

$$M_x = \frac{W \cdot b}{L} \times x - W(x-a).$$

The B.M for all the sections of the beam can be expressed in single equation as

$$M_x = \frac{W \cdot b}{L} \times x \;\vdots\; - W(x-a).$$

The B.M at any section is given by $M = EI\frac{d^2y}{dx^2}$

$$EI\frac{d^2y}{dx^2} = \frac{W \cdot b}{L} \times x \;\vdots\; - W(x-a)$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = \frac{W \cdot b}{L} \cdot \frac{x^2}{2} + C_1 \;\vdots\; - \frac{W(x-a)^2}{2}$$

Integrating above equation once again, we get

$$EI\,y = \frac{W \cdot b}{2L}\,\frac{x^3}{3} + C_1 x + C_2 \;\vdots\; - \frac{W}{2}\,\frac{(x-a)^3}{3}$$

The values of C_1 & C_2 are obtained from boundary conditions

The two boundary conditions are

i) At $x=0$; $y=0$ ii) At $x=L$; $y=0$

i) at $x=0$; $y=0$

$$\Rightarrow \quad EI(0) = \frac{W \cdot b}{2L}(0) + C_1(0) + C_2$$

$$\boxed{C_2 = 0}$$

ii) at B, $x=L$; $y=0$

$$EI(0) = \frac{W \cdot b}{2L} \cdot \frac{L^3}{3} + C_1 \cdot (L) + 0 - \frac{W}{2}\frac{(L-a)^3}{3}$$

$$= \frac{WbL^2}{6} + C_1 L - \frac{Wb^3}{6}$$

$$C_1 L = \frac{W}{6}b^3 - \frac{WbL^2}{6} = -\frac{Wb}{6}\left[L^2 - b^2\right]$$

$$\boxed{C_1 = -\frac{Wb}{6L}\left[L^2 - b^2\right]}$$

Substituting the value of C_1 in Slope we get

$$EI\frac{dy}{dx} = \frac{W \cdot b \cdot x^2}{2L} + \left[-\frac{Wb}{6L}(L^2 - b^2)\right] \vdots - \frac{W(x-a)^2}{2}$$

$$\boxed{EI\,\theta = \frac{W \cdot b \cdot x^2}{2L} - \frac{W \cdot b}{6L}(L^2 - b^2) \vdots - \frac{W(x-a)^2}{2}}$$

The above equation gives slope at any point in the beam.

Slope at A ; $x=0$

$$EI\,\theta_A = \frac{W \cdot b}{2L}(0) - \frac{W \cdot b}{6L}(L^2 - b^2)$$

$$= -\frac{W \cdot b}{6L}(L^2 - b^2)$$

$$\boxed{\theta_A = -\frac{Wb}{6EIL}(L^2 - b^2)}$$

Slope at B; $x=L$

$$EI\,\theta_B = \frac{W \cdot b}{2L}L^2 - \frac{Wb}{6L}(L^2 - b^2) - \frac{W(L-a)^2}{2}$$

$$EI\,\theta_B = \frac{WbL}{2} - \frac{Wb}{6L}(L^2-b^2) - \frac{W(b)^2}{2}$$

$$\theta_B = \frac{Wb}{2EI}\left[L - \frac{(L^2-b^2)}{3L} - b\right].$$

$$\theta_B = \frac{Wb}{2EI}\left[\frac{3aL-(L^2-b^2)}{3L}\right]$$

$$\boxed{\theta_B = \frac{Wb}{6EIL}\left[3a^2+3ab-L^2+b^2\right]}$$

Substitute the values of C_1 & C_2 in deflection, we get

$$\boxed{EI\cdot y = \frac{W\cdot b}{6L}\cdot x^3 + \left[-\frac{Wb}{6L}(L^2-b^2)\right]x + 0 \;\vdots\; -\frac{W}{6}(x-a)^3}$$

equation gives the deflection at any point in the beam.

To find the deflection at 'c' ; $x = a$

$$EI\cdot y_c = \frac{W\cdot b}{6L}(a)^3 - \frac{Wb}{6L}(L^2-b^2)(a)$$

$$= \frac{W\cdot b}{6L}\cdot a\left[a^2 - L^2 + b^2\right]$$

$$= -\frac{W\cdot b\cdot a}{6L}\left[L^2 - a^2 - b^2\right].$$

$$= -\frac{W\cdot a\cdot b}{6L}\left[(a+b)^2 - a^2 - b^2\right]$$

$$= -\frac{W\cdot a\cdot b}{6L}\left[\cancel{a^2}+\cancel{b^2}+2ab-\cancel{a^2}-\cancel{b^2}\right]$$

$$= -\frac{W a\cdot b}{6L}\times 2\cdot a\cdot b$$

$$EI\,y_c = -\frac{W a^2 b^2}{3L}$$

$$\boxed{y_c = -\frac{W a^2 b^2}{3EIL}}$$ *

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Q: A beam of length 6m is supported simply at its ends and Carries two points loads of 48 kN and 40 kN. at a distance of 1m and 3m respectively from the left support. find.

i, deflection under each load

ii, maximum deflection, and

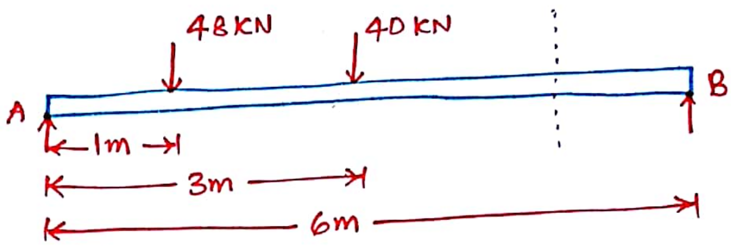
iii, the point at which maximum deflection occurs.

Given, $E = 2\times10^5\ N/mm^2$ and $I = 85\times10^6\ mm^4$.

Sol: Given

$I = 85\times10^6\ mm^4$

$E = 2\times10^5\ N/mm^2$.



Calculate the reactions at R_A & R_B.

Taking moments about 'A'

$$R_B\times6 = 48\times1 + 40\times3 = 168$$

$$R_B = \frac{168}{6} = 28\ kN.$$

$$R_A = \text{Total load} - R_B = (48+40)-28 = 60\ kN.$$

Consider the section 'X' in the last part of the beam at a distance 'x'.

BM at this section is given by.

$$EI\frac{d^2y}{dx^2} = R_A\cdot x \;\vdots\; -48(x-1) \;\vdots\; -40(x-3)$$

$$= 60x \;\vdots\; -48(x-1) \;\vdots\; -40(x-3)$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = 60\frac{x^2}{2} + C_1 \;\vdots\; -48\frac{(x-1)^2}{2} \;\vdots\; -40\frac{(x-3)^2}{2}$$

$$= 30x^2 + C_1 \;\vdots\; -24(x-1)^2 \;\vdots\; -20(x-3)^2$$

Integrating the above equation again, we get.

$$EI.\,y = \frac{30x^3}{3} + C_1x + C_2 \;\vdots\; -24\frac{(x-1)^3}{3} \;\vdots\; -20\frac{(x-3)^3}{3}$$

$$= 10x^3 + C_1x + C_2 \;\vdots\; -8(x-1)^3 \;\vdots\; -\frac{20}{3}(x-3)^3.$$

To find the value of C_1 & C_2, use two boundary conditions

i, at $x = 0$; $y = 0$ and ii, at $x = 6m$; $y = 0$

Substituting first boundary conditions

$$0 = 0 + 0 + C_2 \Rightarrow \therefore \boxed{C_2 = 0.}$$

Substituting second boundary condition:

$$0 = 10 \times 6^3 + C_1 \times 6 + 0 - 8(6-1)^3 - \frac{20}{3}(6-3)^3$$

$$0 = 2160 + 6C_1 - 8 \times 5^3 - \frac{20}{3} \times 3^3$$

$$= 980 + 6C_1$$

$$C_1 = \frac{-980}{6} \Rightarrow \boxed{C_1 = -163.33}$$

Now substituting the values of C_1 & C_2

$$\boxed{EI\,y = 10x^3 - 163.33x \;\vdots\; -8(x-1)^3 \;\vdots\; -\frac{20}{3}(x-3)^3.}$$

i, Deflection under first load:

at point C, obtained by substituting $x = 1m$ upto first dotted line

$$EI \cdot y_C = 10 \times 1^3 - 163.33 \times 1$$

$$= -153.33 \text{ kNm}^3$$

$$= -153.33 \times 10^3 \times 10^9 \text{ Nmm}^3$$

$$y_C = \frac{-153.33 \times 10^{12}}{EI}$$

$$= \frac{-153.33 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} \text{ mm}$$

$$\boxed{y_C = -9.019 \text{ mm}}$$ (-ve represents downward deflection).

ii, Deflection under second load:

at point D, obtained by substituting $x = 3m$

$$EI \cdot y_D = 10 \times 3^3 - 163.33 \times 3 - 8(3-1)^3$$

$$= 270 - 489.99 - 64$$

$$= -283.99 \times 10^{12} \text{ Nmm}^3$$

$$y_D = \frac{-283.99 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6}$$

$$\boxed{y_D = -16.7 \text{ mm}}$$

iii) maximum deflection:

The deflection is likely to be maximum at sctn b/w C & D.

for max. deflection, $\frac{dy}{dx} = 0$.

$$30x^2 + C_1 - 24(x-1)^2 = 0$$
$$30x^2 - 163.33 - 24(x^2 + 1 - 2x) = 0$$
$$6x^2 + 48x - 187.33 = 0.$$

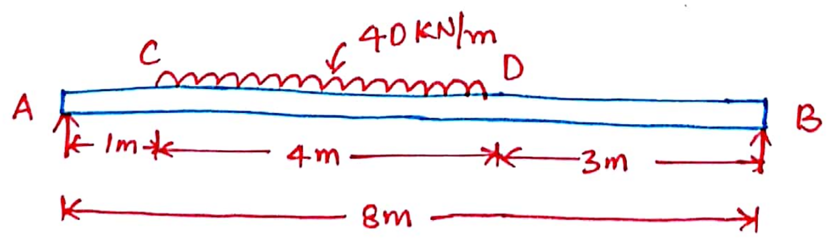
Solution for above quadratic equation

$$x = \frac{-48 \pm \sqrt{48^2 + 4 \times 6 \times 187.33}}{2 \times 6}$$
$$= 2.87 \text{ m}$$

Substituting $x = 2.87$ m in equation for max 'y'

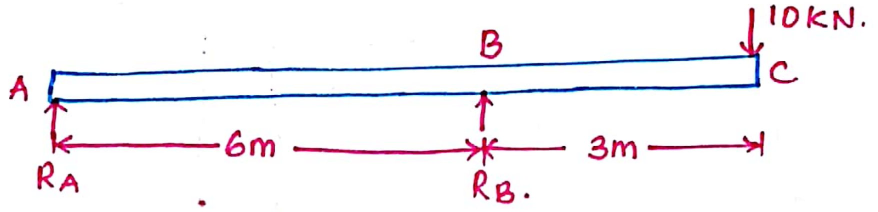
$$EI\, y_{max} = 10 \times 2.87^3 - 163.33 \times 2.87 - 8(2.87-1)^3$$
$$= 236.39 - 468.75 - 52.31$$
$$= -284.67 \times 10^{12} \text{ Nmm}^3$$
$$y_{max} = \frac{-284.67 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6}$$
$$\boxed{y_{max} = -16.745 \text{ mm}}$$

Q: A beam of length 8m is simply supported at its ends. It carries a U.D.L of 40 KN/m as shown in figure. Determine the deflection of the beam at its mid span and also the position of maximum deflection and maximum deflection. Take $E = 2 \times 10^5$ N/mm^2 and $I = 4.3 \times 10^8$ mm^4



Q An Over hanging beam ABC is loaded as shown in figure. Find the slopes over each support and at the right end. Find also the maximum upward deflection between the supports and the deflection at the right end.

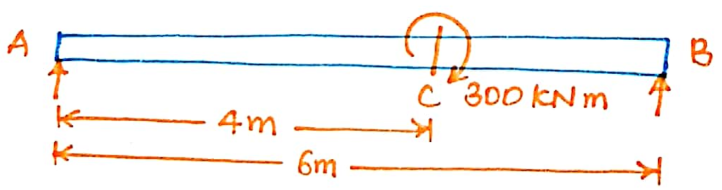
Take $E = 2 \times 10^5 \ N/mm^2$ and $I = 5 \times 10^8 \ mm^4$



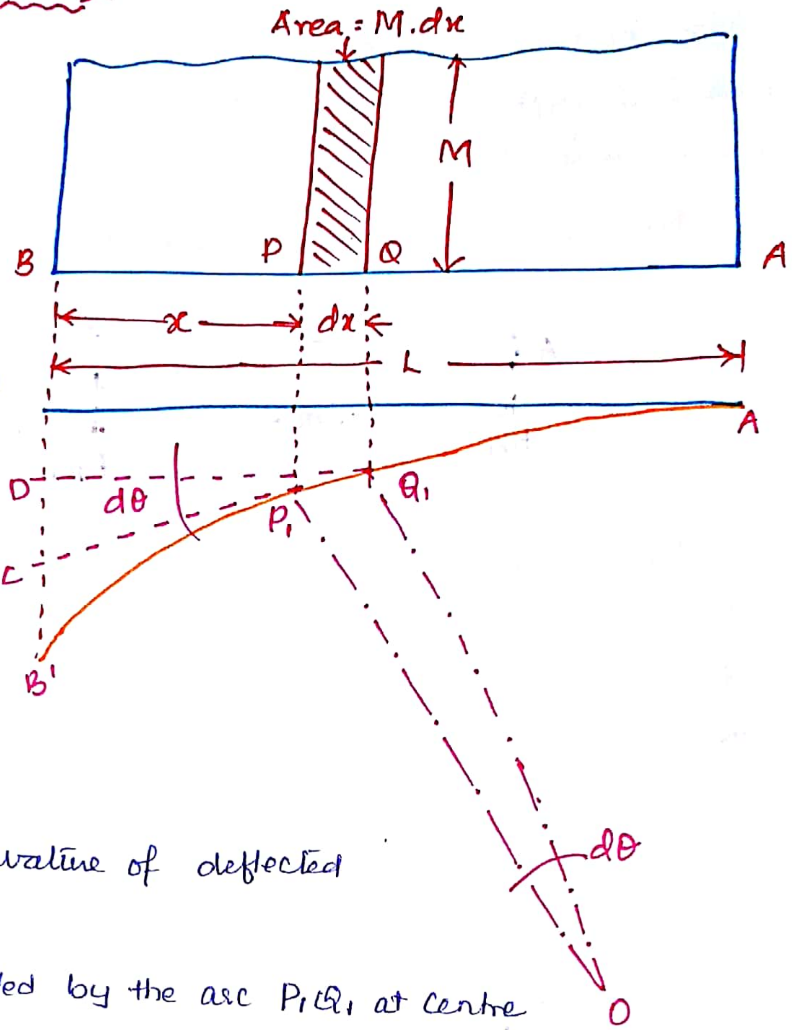
Q: A Horizontal beam AB is simply supported at A and B, 6m apart. The beam is subjected to a clockwise couple of 300 KNm at a distance of 4m from the left end as shown in figure. If $E = 2 \times 10^5 \ N/mm^2$ and $I = 2 \times 10^8 \ mm^4$, determine.

i, deflection at the point where couple is acting.

ii, The maximum deflection.



7. Moment Area Method:-



Consider a beam AB carrying some loading. Let the beam bent into AQ_1P_1B as shown.

Due to the load acting on beam. let A be the point of zero slope and zero deflection.

Consider a element PQ of small length 'dx' at a distance 'x' from B.

Let R – Radius of curvature of deflected part P_1Q_1

$d\theta$ – Angle subtended by the arc P_1Q_1 at centre

M – Bending moment between P and Q.

P_1C – Tangent at Point P_1

Q_1D – Tangent at Point Q_1.

The angle b/w lines CP_1 and DQ_1 will be equal to '$d\theta$'

for the deflected part P_1Q_1 of beam we have

$$P_1Q_1 = R \cdot d\theta$$

$$P_1Q_1 \approx dx$$

$$dx = R \cdot d\theta$$

$$\boxed{d\theta = \frac{dx}{R}} \quad \text{— i,}$$

But we know

$$\frac{M}{I} = \frac{E}{R} \Rightarrow \boxed{R = \frac{EI}{M}}$$

Substitute 'R' in $d\theta$ we get

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$$d\theta = \frac{dx}{\left(\frac{EI}{M}\right)} = \frac{M.dx}{EI.}$$

Since the slope at a point 'A' is assumed zero, total slope at 'B' is obtained by integrating the above equation between the limits 0 & L.

$$\theta = \int_0^L \frac{M.dx}{EI} = \frac{1}{EI}\int_0^L M dx.$$

'M' represents the area of B.M diagram of length 'dx'

Hence $\int_0^L M.dx$ represents the area of B.M diagram b/w A & B.

$$\theta = \frac{1}{EI} \times \text{Area of B.M diagram b/w A \& B.}$$

∴ Slope at 'B'

$$\boxed{\theta_B = \frac{\text{Area of B.M diagram b/w A \& B}}{EI.}}$$

Now the deflection, due to Bending of the portion P_1Q_1 is given by

$$dy = x.d\theta$$

$$dy = x.\frac{M.dx}{EI.}$$

Since deflection @A is zero, hence the total deflection at 'B' is obtained by integrating the above equation b/w limits 0 to L.

$$y = \int_0^L \frac{xM.dx}{EI}$$

$$= \frac{1}{EI}\int_0^L x.M dx.$$

But $x \times M dx$ represents the moment of area of the B.M diagram of length 'dx' about point 'B'.

This is equal to the total area of B.M multiplied by the distance of C.G of the B.M diagram

$$\boxed{y = \frac{1}{EI} \times A \times \bar{x}}$$

A - Area of B.M diagram b/w A & B.

$\bar{x}$ - Distance of C.G of Area from A to B.

8. MOHR'S THEOREMS :-

The results given for slope and deflection in moment area method are known as "Mohr's theorems"

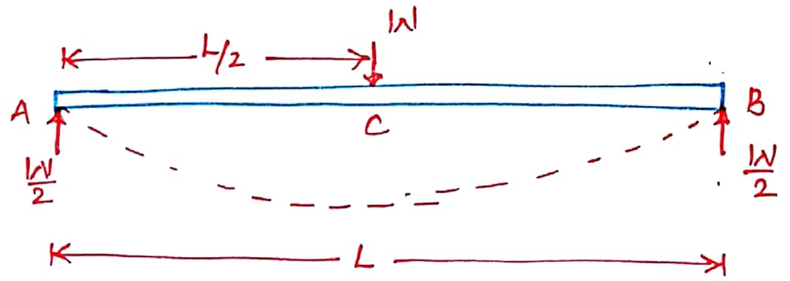
I. The change of slope between any two points is equal to the net area of the B.M diagram between these points divided by EI.

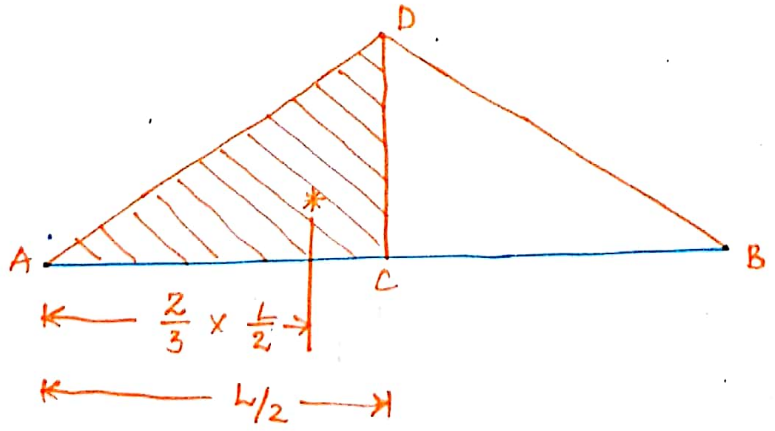
II. The total deflection between any two points is equal to the moment of the area of BM diagram between the two points about the last point divided by EI.

The Mohr's theorems is conveniently used for following cases

1. Problems on Cantilevers (zero slope at fixed end).
2. Simply supported beams carrying symmetrical loading (zero slope at the centre).
3. Beams fixed at both ends (zero slope at each end).

⊛ SLOPE & DEFLECTION OF A SIMPLY SUPPORTED BEAM CARRYING A POINT LOAD AT THE CENTRE BY MOHR'S THEOREM:





Consider a simply supported beam AB of length 'L' and carrying a point load 'W' at the centre of the beam i.e., at 'C'.

$\because$ This is a case of symmetrical loading, hence slope is zero at the centre.

Now, using Mohr's theorem for slope

Slope at 'A'

$$A = \frac{\text{Area of B.M diagram b/w A \& C}}{EI}$$

Area of B.M diagram b/w A & C

$$= \text{Area of } \Delta^{le} \text{ ACD}$$

$$= \frac{1}{2} \times \frac{L}{2} \times \frac{WL}{4} = \frac{WL^2}{16}$$

$\therefore$ Slope at A θ $\boxed{\theta_A = \frac{WL^2}{16EI}}$

Now using Mohr's theorem for deflection, we get

$$y = \frac{A\bar{x}}{EI}.$$

A - Area of BM diagram b/w A & C

$$= \frac{WL^2}{16}$$

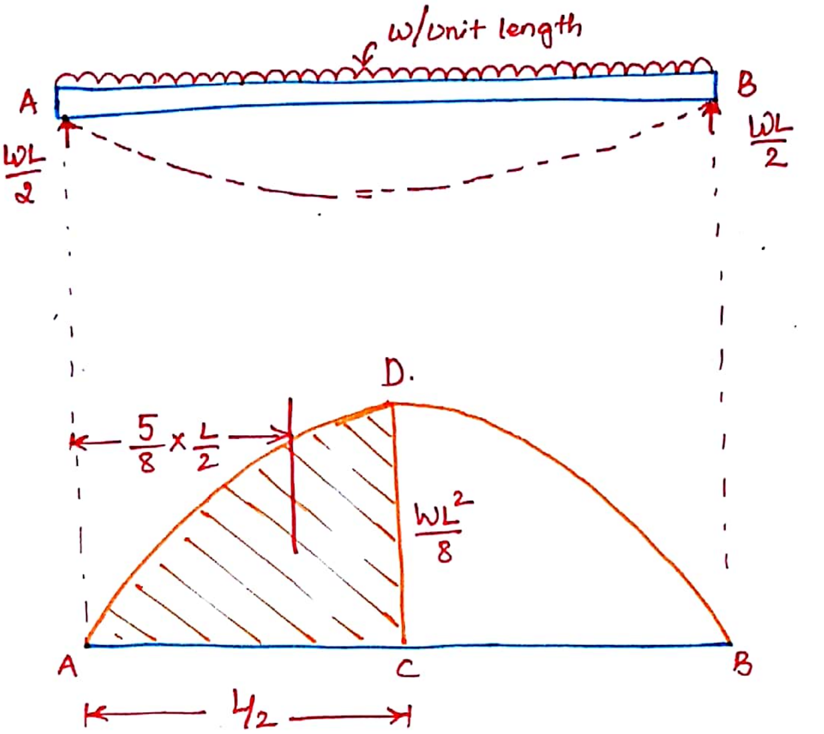
$\bar{x}$ - Distance of C.G of area 'A' from A

$$= \frac{2}{3} \times \frac{L}{2} = \frac{L}{3}$$

$$\therefore y = \frac{\frac{WL^2}{16} \times \frac{L}{3}}{EI}$$

$$\boxed{y = \frac{WL^3}{48EI}.}$$

* SLOPE & DEFLECTION OF A SIMPLY SUPPORTED BEAM CARRYING U.D.L BY MOHR'S THEOREM:



$$\text{Slope at } A = \frac{\text{Area of B.M diagram b/w A and C.}}{EI}$$

$$\text{Area} = \text{Area of parabola } ACD$$

$$= \frac{2}{3} \times AC \times CD = \frac{2}{3} \times \frac{L}{2} \times \frac{wL^2}{8} = \frac{w \cdot L^3}{24}$$

$$\therefore \boxed{\theta_A = \frac{wL^3}{24EI}}$$

for deflection $\bar{x}$ = Distance of C.G of area

$$= \frac{5}{8} \times AC = \frac{5}{8} \times \frac{L}{2} = \frac{5L}{16}$$

$$y = \frac{\frac{wL^3}{24} \times \frac{5L}{16}}{EI} \qquad \boxed{y = \frac{5}{384} \frac{w \cdot L^4}{EI}.}$$

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9. CONJUGATE BEAM METHOD:-

The slope and deflections of beams and cantilevers obtained from various methods become laborious, when applied to beams whose flexural rigidity is not uniform throughout the length of the beam.

The slopes and deflections of such beams can be easily obtained by Conjugate beam method.

CONJUGATE BEAM:-

It is an imaginary beam of length equal to that of the original beam but for which the load diagram is the $\frac{M}{EI}$ diagram.

[The load at any point on the conjugate beam is equal to BM at that point divided by EI].

*1. The Slope at any section of the given beam is equal to the "Shear force" at the corresponding section of the "Conjugate beam".

*2. The deflection at any section for the given beam is equal to the "Bending moment" at the corresponding section of the "Conjugate beam".

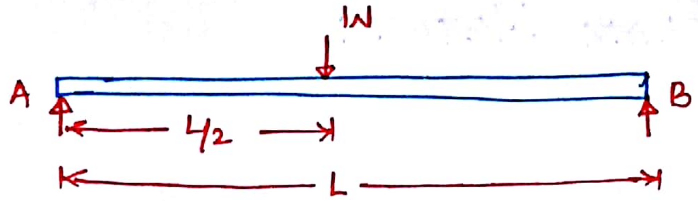
10. SLOPE & DEFLECTION OF A S.S. BEAM WITH A POINT LOAD AT CENTRE:

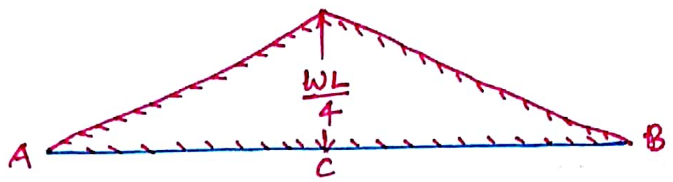
(•) Consider a simply supported beam AB of length 'L' carrying a point load 'W' at the Centre 'C'

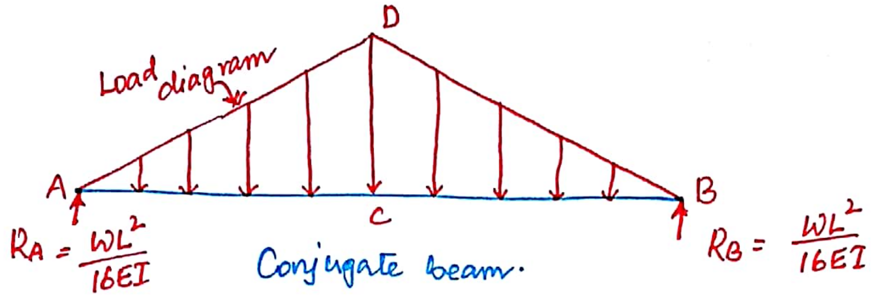
(•) The B·M at A and B is Zero and at the Centre B·M will be $\frac{W \cdot L}{4}$.

(•) The B·M varies according to Straight line law.

(•) The load on the conjugate beam will be obtained by dividing the B·M at that point by EI.







Total load on the conjugate beam = Area of the load diagram

$$= \frac{1}{2} \times AB \times CD$$

$$= \frac{1}{2} \times L \times \frac{WL}{4EI}.$$

$$= \frac{WL^2}{8EI}$$

Reaction at each support for the conjugate beam will be half of the total load.

$$R_A^* = R_B^* = \frac{1}{2}\,\frac{WL^2}{8EI} = \frac{WL^2}{16EI}$$

According to Conjugate Beam method:

θ_A = Shear force at 'A' for the conjugate beam.

$$= R_A^*$$

$$\boxed{\theta_A = \frac{WL^2}{16EI}}$$

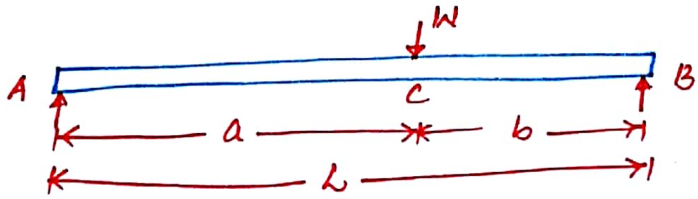
y_c = B.M at C for the conjugate beam

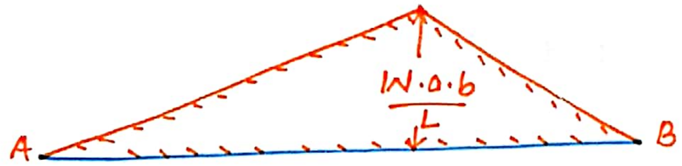
$= R_A^* \times \frac{L}{2}$ − Load corresponding to ACD.

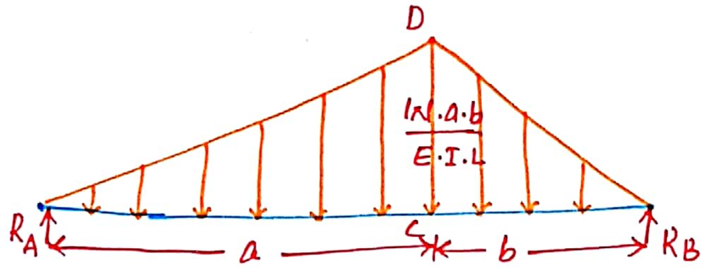
$$= \frac{WL^2}{16EI} \times \frac{L}{2} - \left(\frac{1}{2} \times \frac{L}{2} \times \frac{WL}{4EI}\right) \times \left(\frac{L}{2} \times \frac{1}{3}\right)$$

$$\boxed{y_c = \frac{WL^3}{48EI}}$$

11. SIMPLY SUPPORTED BEAM CARRYING AN ECCENTRIC LOAD:







Reactions acting at A and B are

$$R_A = \frac{W.b}{L} \quad ; \quad R_B = \frac{W.a}{L}$$

The B.M at A and B will be zero; BM at 'C' will be $R_A \times a$

$$= \frac{W \times b}{L} \times a = \frac{W.a.b}{L}$$

The vertical load on conjugate beam at C^* will be $\frac{W.a.b}{EI.L}$

To find reactions at A and B

Taking moments about A

$$R_B^* . L = \left[\frac{1}{2} \times AC \times CD\right] \times \left(\frac{2}{3} \times AC\right) + \left(\frac{1}{2} \times BC \times CD\right)\left[AC + \frac{1}{3} CB\right]$$

$$= \frac{1}{2} \times a \times \frac{W.ab}{EIL} \times \frac{2}{3} \times a + \frac{1}{2} \times b \times \frac{W.a.b}{EIL} \times \left(a + \frac{b}{3}\right)$$

$$= \frac{W.a.b}{2EIL}\left[a \times \frac{2}{3} \times a + ba + \frac{b^2}{3}\right]$$

$$= \frac{W.a.b}{2EIL}\left[\frac{2a^2 + 3ab + b^2}{3}\right]$$

$$= \frac{W.a.b}{6EIL}\left[(a+b)^2 + a^2 + ab\right]$$

$$= \frac{W.a.b}{6EIL}\left[L^2 + a(a+b)\right]$$

$$= \frac{W.a.b}{6EIL}\left[L^2 + aL\right]$$

$$= \frac{W \cdot a \cdot b}{6EIL} [L^2 + a \cdot L]$$

$$R_B \times L = \frac{W \cdot a \cdot (L-a)}{6EI\cancel{L}} \times \cancel{L} (L+a)$$

$$\boxed{R_B = \frac{W \cdot a}{6EIL} [L^2 - a^2]}$$

Similarly

$$\boxed{R_A = \frac{W \cdot b}{6EIL} [L^2 - b^2]}$$

let θ_A = Slope at A & y_c - Deflection at 'c'

Now, According to Conjugate beam method

θ_A = Shear force at 'A' for the Conjugate beam

$= R_A$

$$* \boxed{\theta_A = \frac{W \cdot b}{6EIL} (L^2 - b^2)} *$$

y_c = B.M at C for Conjugate beam

$= R_A^* \cdot a$ - Load ACD x Distance of C.G of ACD from C

$$= \frac{W \cdot b}{6EIL} (L^2 - b^2) \times a - \left(\frac{1}{2} \times a \times \frac{W \cdot a \cdot b}{EIL}\right) \times \left(\frac{a}{3}\right)$$

$$= \frac{W \cdot a \cdot b}{6EIL} (L^2 - b^2) - \frac{W \cdot a^3 \cdot b}{6EIL}$$

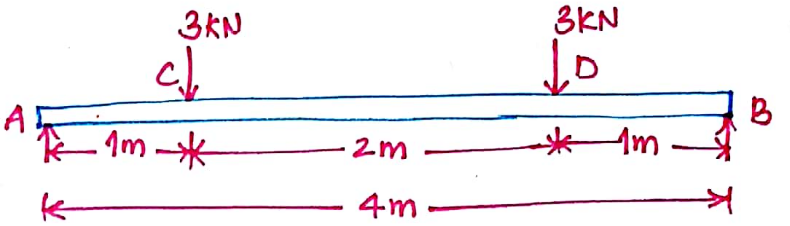
$$* \boxed{y_c = \frac{W \cdot a \cdot b}{6 \cdot EIL} [L^2 - b^2 - a^2].} *$$

Q. A simply supported beam of length 4m carries a point load of 3KN at a distance of 1m from each end. If $E = 2\times10^5\ N/mm^2$ and $I = 10^8\ mm^4$ for the beam, then using conjugate beam method, determine

i, Slope at each end and under each load

ii, deflection under each load and at the centre.

Sol:



Given

length, $L = 4m$

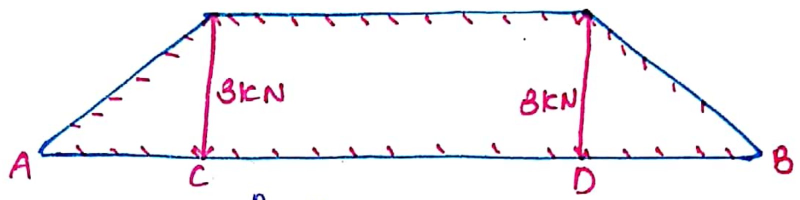
$E = 2\times10^5\ N/mm^2$

$= \frac{2\times10^5\times10^{-3}}{10^{-6}}\ \frac{KN}{m^2}$

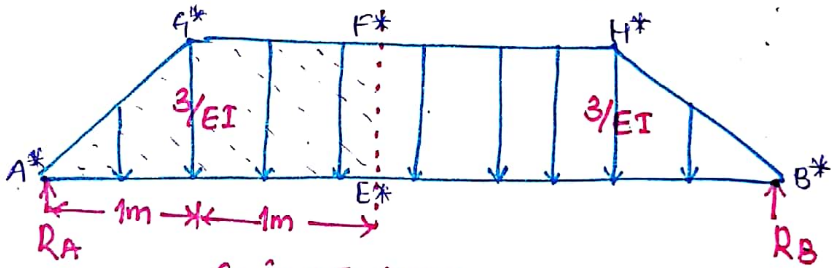
$= 2\times10^8\ KN/m^2$

$I = 10^8\ mm^4 = \frac{10^8}{10^{12}}\ m^4$

$= 10^{-4}\ m^4$



Bending moment diagram.



Conjugate beam.

Reactions at A & B are

$R_A = R_B = 3KN$

B.M @ A & B = 0

BM @ C = $R_A \times 1 = 3\times1 = 3KN$

BM @ D = $R_B\times1 = 3\times1 = 3KN$ (or), $R_A\times3 - R\times2 = 9\times3 - 3\times2 = 3\ KN$.

From the obtained Bending moment dividing at any section by EI, we can construct conjugate beam.

let R_A^* & R_B^* = Reactions of Conjugate Beam.

$R_A^* = R_B^*$ = Half of total load on conjugate Beam.

$= \frac{1}{2}\left[A^*B^*H^*G^*\right]$

$= \frac{1}{2}\left[\frac{(4+2)}{2} * \frac{3}{EI}\right]$

$R_A^* = R_B^* = \frac{4.5}{EI}$.

i, Slope at each end and under each load:-

Slope at A, θ_A = Shear force at A for conjugate beam

$$= R_A^* = \frac{4.5}{EI} = \frac{4.5}{2\times10^8\times10^{-4}}$$

$$= 0.000225 \text{ rad}.$$

$$\theta_B = R_B^* = \frac{4.5}{EI} = 0.000225 \text{ rad}$$

θ_C = Shear force at 'C' for conjugate beam

= R_A^* − Total load of Δle

$$= \frac{4.5}{EI} - \frac{1}{2}\times1\times\frac{3}{EI} = \frac{3}{EI} = \frac{3}{2\times10^8\times10^{-4}}$$

$$= 0.00015 \text{ rad}.$$

Similarly, θ_D = 0.00015 rad.

ii, Deflection under each load:-

let y_C & y_D are deflections at 'C' & 'D' for the beam

Now according to conjugate beam

y_C = B.M at C^* for conjugate beam

= $R_A^* \times 1$ − ($\frac{1}{2}$ Triangle load) × Distance of C.G

$$= \frac{4.5}{EI}\times1 - \left(\frac{1}{2}\times1\times\frac{3}{EI}\right)\times\frac{1}{3}$$

$$= \frac{4.0}{EI} = \frac{4.0}{2\times10^8\times10^{-4}} \text{ m} = \frac{4\times1000}{2\times10^4} \text{ mm}$$

$$= \underline{0.2 \text{ mm}}$$

Deflection at the centre of the beam

= B.M at the centre of the conjugate beam.

= $R_A^* \times 2$ − load $A^*E^*F^*$ × Distance of C.G

$$= \frac{4.5}{EI}\times2 - \left(\frac{1}{2}\times1\times\frac{3}{EI}\right)\times\left(1+\frac{1}{3}\right) - \left(1\times\frac{3}{EI}\right)\times\frac{1}{2}$$

$$= \frac{9}{EI} - \frac{2}{EI} - \frac{1.5}{EI} = \frac{6.5}{EI}$$

$$= \frac{6.5}{2\times10^8\times10^{-4}} \text{ m} = \frac{6.5\times1000}{2\times10^4} = \underline{0.325 \text{ mm}}$$